POINCARÉ AND LOGARITHMIC SOBOLEV CONSTANTS FOR METASTABLE MARKOV CHAINS VIA CAPACITARY INEQUALITIES

ANDRÉ SCHLICHTING AND MARTIN SLOWIK

ABSTRACT. We investigate the metastable behaviour of reversible Markov chains on countable infinite state spaces. Based on a definition of metastable sets, we compute precisely the mean exit time from a metastable set. Under additional size and regularity properties of metastable sets, we establish asymptotic sharp estimates on the Poincaré and logarithmic Sobolev constant. The main ingredient are capacitary inequalities along the lines of V. Maz'ya relating regularity properties of harmonic functions and capacities. We exemplify the usefulness of our approach in the context of the random field Curie-Weiss model.

CONTENTS

1. Introduction 2
1.1. Setting 5
1.2. Main result 7
1.3. Random field Curie-Weiss model 10
2. Functional inequalities 13
2.1. Capacitary inequality 14
2.2. Orlicz-Birnbaum estimates 15
2.3. Poincaré and Sobolev inequalities 17
3. Application to metastable Markov chains 21
3.1. A priori estimates 21
3.2. Poincaré inequality 24
3.3. Logarithmic Sobolev inequality 27
4. Random field Curie-Weiss model 30
4.1. Verification of ϱ-metastability 30
4.2. Regularity estimates via coupling arguments 33
4.3. Local mixing estimates within metastable sets 37
Appendix A. Young functions 40
Acknowledgement 41
References 41

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1. INTRODUCTION

Metastability is a *dynamical* phenomenon that is characterized by the existence of multiple, well separated time scales. Depending on the time scales under consideration, the state space can be decomposed into several disjoint subsets (*metastable partition*) with the property that typical transition times between different subsets are long compared to characteristic mixing times within each subset.

For a rigorous mathematical analysis of metastable Markov processes various different methods has been invented. The *pathwise approach* [18, 38] based on large deviation methods in path space [23] has been proven to be robust and rather universal applicable. While it yields detailed information e.g. in the typical exit path, its precision to predict quantities of interest like the mean transition time is, however, limited to logarithmic equivalence. For reversible systems the *potential theoretic approach* [12, 14, 11] has been developed to establish sharp estimates on the mean transition time and the low-lying eigenvalues and to prove that the transition times are asymptotically exponential distributed. A key ingredient of this concept is to express probabilistic quantities of interest in terms of capacities and to use variational principles to compute the later. For metastable Markov processes in which the expected transition times for a large number of subsets is of the same order *martingale approach* [3] has been recently developed to identify the limiting process on the time scale of the expected transition times as an Markov process via the solution of a martingale problem.

In the context of Markov processes, there is also spectral signature of metastability. Since the transition probabilities between different subsets of the metastable partition are extremely small, an irreducible Markov process exhibiting a metastable behavior can be seen as a perturbation of the reducible version of it in which transitions between different subsets of the metastable partition are forbidden. For the reducible version, the theorem of Perron-Frobenius implies that the eigenvalue zero of the associated generator is degenerate with multiplicity given by the number of set in the metastable partition. In particular, the corresponding eigenfunctions are given as indicator functions on these subsets. Provided that the perturbation is sufficiently small, this leads typically to a clustering of small eigenvalues separated by a gap from the rest of the spectrum.

The main objective of the present work is to extend the potential theoretic approach for deriving sharp asymptotics of the spectral gap and the logarithmic Sobolev constants for metastable Markov chains on countable infinite state spaces.

So far sharp estimates of low-lying eigenvalues has been derived in the following settings:

(1) For a class of reversible Markov processes on discrete state spaces that are strongly recurrent in the sense that within each set of the metastable partition there is at least one single point that the process visits with overwhelming probability before leaving the corresponding set of the metastable

partition. Based on the potential theoretic approach sharp estimates on the low-lying eigenvalues and associated eigenfunctions has been obtained under some additional non-degeneracy conditions in [13]. Typical examples of strongly recurrent Markov chains are finite-state Markov processes with exponential small transition probabilities [4] and models from statistical mechanics under either Glauber or Kawasaki dynamics in finite volume at very low temperature [11, 16].

(2) For reversible diffusion processes in a potential landscape in $\mathbb{R}^d$ subject to small noise sharp estimates on the low-lying eigenvalues have been obtained in [15, 43]. The proof relies on potential theory and a priori regularity estimates of solutions to certain boundary value problems. Based on hypoelliptic techniques and a microlocal analysis of the corresponding Witten-complex a complete asymptotic expansion of the lowest eigenvalues was shown in [24]. Recently, based on methods of optimal transport, an interesting approach to derive a sharp characterization of the Poincaré and logarithmic Sobolev constants has been developed in [35, 40].

A common starting point for mathematical rigorous investigations in the settings describe above is the identification of a set of *metastable points* that serves as representatives of the sets in the metastable partition. For strongly recurrent Markov chains the set of metastable points $\mathscr{M}$ is chosen in such a way that, for each $m \in \mathscr{M}$, the probability to escape from m to the remaining metastable points $\mathscr{M} \setminus \{m\}$ is small compared with the probability to reach $\mathscr{M}$ starting at some arbitrary point in the state space before returning to it, cf. [11, Definition 8.2]. In the context of reversible diffusion processes, metastable points are easy to identify and correspond to local minima of the potential landscape. Since in dimensions $d > 1$ diffusion processes do not hit individual points $x \in \mathbb{R}^d$ in finite time, each metastable point has to be enlarged, cf. [10, Definition 8.1]. For instance, each metastable point $m \in \mathscr{M}$ can be replaced by a small ball $B_\varepsilon(m)$. The radius $\varepsilon > 0$ of such balls should be chosen large enough to ensure that it is sufficiently likely for the process to hit $B_\varepsilon(m)$, but simultaneously small enough to control typical oscillations of harmonic functions within these balls.

Once the set of metastable points has been identified, the low-lying eigenvalues are also characterized in terms of mean exit times for generic situations. Namely, each low-lying eigenvalue is equal to the inverse of the mean exit time from the corresponding metastable point up to negligible error terms.

Starting ideas. One would expect that the strategy of enlargements of metastable points that has been successfully used in the diffusion setting, should also apply to stochastic spin systems at finite temperature or in growing volumes. However, proving general regularity estimates for solutions of elliptic equations is challenging on high dimensional discrete spaces and, so far, highly model dependent.

The present work provides a mathematical definition of metastability for Markov chains on *countable infinite* state spaces, see Definition 1.1, where the metastable points that represents the sets in the metastable partition are replaced by metastable sets. An advantage of this definition is that one can immediately deduce sharp estimates on the mean exit time to "deeper" metastable sets without using additional regularity estimates of harmonic functions, cf. Proposition 1.12. Moreover, sharp estimates on the smallest non-zero eigenvalues of the generator follow under the natural assumption of good mixing properties within metastable sets and some rough estimates on the regularity of the harmonic function at the boundary of metastable sets. The main tool in the proof is the *capacitary inequality*, see Theorem 2.1.

A key observation leading to the present definition of metastability is the following: It is well know that classical Poincaré–Sobolev inequalities on $\mathbb{Z}^d$ for functions with compact support, say on a ball $B_r(x) \subset \mathbb{Z}^d$, follow from isoperimetric properties of the underlying Euclidean space and the so-called co-area formula. The necessary isoperimetric inequality states that

$$|A|^{(d-1)/d} \leq C_{\mathrm{iso}}\, |\partial A|, \qquad \forall\, A \subset B_r(x),$$

where $|A|$ and ∂A denotes the cardinality and the boundary of the set A. The latter is defined as the set of all points $x \in A$ for which there exists a $y \notin A$ such that $\{x, y\}$ is an element of the edge set of $\mathbb{Z}^d$. For a positive recurrent Markov chain with state space $\mathscr{S}$ and invariant distribution μ functional inequalities can also be established provided that the isoperimetric inequality is replaced by a measure-capacity inequality, cf. Proposition 2.5. For $B \subset \mathscr{S}$ and $\Psi \colon \mathbb{R}_+ \to \mathbb{R}_+$ being a convex function, the measure-capacity inequality is given by

$$\mu[A]\, \Psi^{-1}\big(1/\mu[A]\big) \leq C_\Psi \operatorname{cap}(A, B^c), \qquad \forall\, A \subset B.$$

Inspired by the form of the measure-capacity inequality, we propose a definition of metastability for Markov chains that in addition encodes local isoperimetric properties by considering for any subset A outside of the union of the metastable sets its escape probability to the union of the metastable sets.

In order to demonstrate the usefulness of our approach, we prove sharp estimates on the spectral gap and the logarithmic Sobolev constants for the *random field Curie-Weiss model* at finite temperature and with a continuous bounded distribution of the random field. In order to prove rough regularity estimates of harmonic functions we use a coupling construction that has been originally invented in [6].

In the present work we decided to focus only on discrete-time Markov chains to keep the presentation as clear as possible. However, our methods also apply to Markov chains in continuous time with obvious modifications.

The remainder of this paper is organized as follows. In the next subsection we describe the setting to which our methods apply. In Subsection 1.2 and 1.3 we state our main results. In Section 2 we first prove the capacitary inequality for reversible Markov processes. In particular, we show how this universal estimate allows to derive so called Orlicz-Birnbaum estimates from which Poincaré and logarithmic Sobolev constants can be deduced. Then, we prove our main results in Section 3.

Finally, in Section 4 we apply the previously developed methods to the random field Curie-Weiss model.

1.1. **Setting.** Consider an *irreducible* and *positive recurrent* Markov process $X = (X(t) : t \geq 0)$ in discrete-time on a countable state space $\mathcal{S}$ with transition probabilities denoted by $(p(x,y) : x,y \in \mathcal{S})$. For any measurable and bounded function $f : \mathcal{S} \to \mathbb{R}$, define the corresponding (discrete) generator by

$$\big(Lf\big)(x) := \sum_{y\in\mathcal{S}} p(x,y)\big(f(y)-f(x)\big). \tag{1.1}$$

Throughout, we assume that the Markov chain is *reversible* with respect to a unique invariant distribution μ. That is, the transitions probabilities satisfy the *detailed balance condition*

$$\mu(x)\,p(x,y) = \mu(y)\,p(y,x) \qquad \text{for all } x,y \in \mathcal{S}. \tag{1.2}$$

We denote by $\mathbb{P}_\nu$ the law of the Markov process given that it starts with initial distribution ν. If the initial distribution is concentrated on a single point $x \in \mathcal{S}$, we simply write $\mathbb{P}_x$. For any $A \subset \mathcal{S}$, let τ_A be the first hitting time of the set A *after* time zero, that is

$$\tau_A := \inf\big\{t > 0 \,:\, X(t) \in A\big\}.$$

So for $X(0) \in A$, τ_A is the *first return time* to A and for $X(0) \notin A$, τ_A is the *first hitting time* of A. In case that the set A is a singleton $\{x\}$ we write τ_x instead of $\tau_{\{x\}}$.

We are interested in Markov chains that exhibits a metastable behavior. For this purpose we introduce the notion of metastable sets.

Definition 1.1 (Metastable sets)**.** For fixed $\varrho > 0$ and $K \in \mathbb{N}$ let $\mathcal{M} = \{M_1,\dots,M_K\}$ be a set of subsets of $\mathcal{S}$ such that $M_i \cap M_j = \emptyset$ for all $i \neq j$. A Markov chain $(X(t) : t \geq 0)$ is called ϱ-metastable with respect to a *set of metastable sets* $\mathcal{M}$, if

$$|\mathcal{M}|\,\frac{\max_{M\in\mathcal{M}} \mathbb{P}_{\mu_M}\big[\tau_{\bigcup_{i=1}^K M_i\setminus M} < \tau_M\big]}{\min_{A\subset\mathcal{S}\setminus\bigcup_{i=1}^K M_i} \mathbb{P}_{\mu_A}\big[\tau_{\bigcup_{i=1}^K M_i} < \tau_A\big]} \leq \varrho \ll 1, \tag{1.3}$$

where $\mu_A[x] = \mu[x \mid A]$, $x \in A \neq \emptyset$ denotes the conditional probability on the set A and $|\mathcal{M}|$ denotes the cardinality K of $\mathcal{M}$.

Remark 1.2. In comparison with [11, Definition 8.2 and Remark 8.3], the major advantage of Definition 1.1 is that it does not depend explicitly on the cardinality of the state space $\mathcal{S}$. Moreover, notice that if $|\mathcal{S}| < \infty$ and $|M_i| = 1$ for all $i = 1,\dots,K$, [11, Equation (8.1.5)] implies (1.3). Indeed, by reversibility we have for any $\emptyset \neq A \subset \mathcal{S} \setminus \mathcal{M}$ that

$$\begin{aligned}\mathbb{P}_{\mu_A}\big[\tau_{\mathcal{M}} < \tau_A\big] &\geq \frac{1}{|A|}\sum_{y\in A}\sum_{m\in\mathcal{M}} \frac{\mu(m)}{\mu[A]}\,\mathbb{P}_m\big[\tau_y < \tau_{\mathcal{M}}\big]\\ &= \frac{1}{|A|}\sum_{y\in A}\frac{\mu(y)}{\mu[A]}\,\mathbb{P}_y\big[\tau_{\mathcal{M}} < \tau_y\big] \geq \frac{1}{|\mathcal{S}|}\min_{y\in\mathcal{S}\setminus\mathcal{M}} \mathbb{P}_y\big[\tau_{\mathcal{M}} < \tau_y\big].\end{aligned}$$

Assumption 1.3. *Assume that for some $2 \leq K < \infty$ there exists non-empty, disjoint subsets $M_1, \ldots, M_K \subset \mathcal{S}$ and $\varrho > 0$ such that the Markov chain $(X(t) : t \geq 0)$ is ϱ-metastable with respect to $\mathcal{M} = \{M_i : i \in 1, \ldots, K\}$.*

The definition of metastable sets induces an almost canonical partition of the state space $\mathcal{S}$ into local valleys.

Definition 1.4 (Metastable partition)**.** For any $M_i \in \mathcal{M}$, the *local valley* $\mathcal{V}_i$ around the metastable set M_i is defined by

$$\mathcal{V}_i := \Big\{x \in \mathcal{S} : \mathbb{P}_x\big[\tau_{M_i} < \tau_{\bigcup_{j=1}^K M_j \setminus M_i}\big] \geq \max_{M' \in \mathcal{M} \setminus M} \mathbb{P}_x\big[\tau_{M'} < \tau_{\bigcup_{j=1}^K M_j \setminus M'}\big]\Big\}.$$

A set of metastable sets $\mathcal{M} = \{M_1, \ldots, M_K\}$ gives rise to a *metastable partition* $\{\mathcal{S}_i : i = 1, \ldots, K\}$ of the state space $\mathcal{S}$, that is

$$\text{(i)} \qquad \mathcal{S}_i \subset \mathcal{V}_i, \qquad\qquad \text{(ii)} \qquad \biguplus_{i=1}^K \mathcal{S}_i = \mathcal{S}$$

Remark 1.5. Notice that, by Lemma 3.2, any point $x \in \mathcal{V}_i \cap \mathcal{V}_j$ that lies in the intersection of two different local valley has a negligible mass compared to the mass of the corresponding metastable sets $\mu[M_i]$ and $\mu[M_j]$.

We simply write $\mu_i[\cdot] := \mu[\,\cdot\,|\mathcal{S}_i]$ to denote the corresponding conditional measure. Let $\ell^2(\mu)$ be the weighted Hilbert space of all square summable functions $f : \mathcal{S} \to \mathbb{R}$ and denote by $\langle \cdot, \cdot \rangle_\mu$ the scalar product in $\ell^2(\mu)$. Due to the detailed balance condition (1.2) the generator, L, is symmetric in $\ell^2(\mu)$, that is $\langle -Lf, g\rangle_\mu = \langle f, -Lg\rangle_\mu$ for any $g, f \in \ell^2(\mu)$. The associated *Dirichlet form* is given for any $f \in \ell^2(\mu)$ by

$$\mathcal{E}(f) := \langle f, -Lf\rangle_\mu = \frac{1}{2} \sum_{x,y \in \mathcal{S}} \mu(x)\, p(x,y) \big(f(x) - f(y)\big)^2,$$

which by the basic estimate $\mathcal{E}(f) \leq \|f\|^2_{\ell^2(\mu)}$ is well-defined. Throughout the sequel, let $A, B \subset \mathcal{S}$ be disjoint and non-empty. The *equilibrium potential*, $h_{A,B}$, of the pair (A, B) is defined as the unique solution of the boundary value problem

$$\begin{cases} (Lf)(x) = 0, & x \in \mathcal{S} \setminus (A \cup B) \\ f(x) = \mathbb{1}_A(x), & x \in A \cup B. \end{cases} \tag{1.4}$$

Note that the equilibrium potential has a natural interpretation in terms of *hitting probabilities*, namely $h_{A,B}(x) = \mathbb{P}_x[\tau_A < \tau_B]$ for all $x \in \mathcal{S} \setminus (A \cup B)$. A related quantity is the *equilibrium measure*, $e_{A,B}$, on A which is defined through

$$e_{A,B}(x) := -\big(Lh_{A,B}\big)(x) = \mathbb{P}_x[\tau_B < \tau_A], \qquad \forall\, x \in A. \tag{1.5}$$

Clearly, the equilibrium measure is only non-vanishing on the (inner) boundary of the sets A and B. Further, the *capacity* of the pair (A, B) with potential one on A and zero on B is defined by

$$\operatorname{cap}(A, B) := \sum_{x \in A} \mu(x)\, e_{A,B}(x) = \sum_{x \in A} \mu(x)\, \mathbb{P}_x[\tau_B < \tau_A] = \mathcal{E}(h_{A,B}). \tag{1.6}$$

In particular, we have that

$$\mathbb{P}_{\mu_A}[\tau_B < \tau_A] \;=\; \frac{\operatorname{cap}(A,B)}{\mu[A]}, \tag{1.7}$$

Moreover, $\operatorname{cap}(A,B) = \operatorname{cap}(B,A)$ and, as an immediate consequence of the probabilistic interpretation of capacities, cf. (1.6), we have that

$$\operatorname{cap}(A,B') \;\leq\; \operatorname{cap}(A,B) \qquad \forall\, B' \subset B. \tag{1.8}$$

Let us emphasize that capacities have several variational characterizations, see [42], which can be used to obtain upper and lower bounds. One of them is the *Dirichlet principle*

$$\operatorname{cap}(A,B) \;=\; \inf\bigl\{\mathscr{E}(f) \,:\, f|_A = 1,\, f|_B = 0,\, 0 \leq f \leq 1\bigr\} \tag{1.9}$$

with $f = h_{A,B}$ as its unique minimizer. Further, we denote by $\nu_{A,B}$ the *last-exit biased distribution* that is defined by

$$\nu_{A,B}(x) \;:=\; \frac{\mu(x)\,\mathbb{P}_x\bigl[\tau_B < \tau_A\bigr]}{\sum_{y\in A}\mu(y)\,\mathbb{P}_y\bigl[\tau_B < \tau_A\bigr]} \;=\; \frac{\mu(x)\,e_{A,B}(x)}{\operatorname{cap}(A,B)}, \qquad \forall\, x \in A. \tag{1.10}$$

Notice that $\nu_{A,B} \ll \mu_A$ for any non-empty, disjoint subsets $A, B \subset \mathscr{S}$.

Remark 1.6. In view of (1.7), the condition (1.3) can alternatively be written as

$$\forall A \subset \mathscr{S} \setminus \bigcup_{i=1}^{K} M_i \;\forall M \in \mathscr{M} : \quad |\mathscr{M}|\, \frac{\operatorname{cap}\bigl(M, \bigcup_{i=1}^{K} M_i \setminus M\bigr)/\mu[M]}{\operatorname{cap}\bigl(A, \bigcup_{i=1}^{K} M_i\bigr)/\mu[A]} \;\leq\; \varrho \;\ll\; 1.$$

Hence, the assumption of metastability is essentially a quantified comparison of capacities and measures.

Finally, we write $\mathrm{E}_\nu[f]$ and $\mathrm{Var}_\nu[f]$ to denote the expectation and the variance of a function $f : \mathscr{S} \to \mathbb{R}$ with respect to a probability measure ν. Moreover, we define the relative entropy by

$$\mathrm{Ent}_\nu[f^2] \;=\; \mathrm{E}_\nu[f^2 \ln f^2] - \mathrm{E}_\nu[f^2]\,\ln \mathrm{E}_\nu[f^2],$$

where we indicate the probability distribution ν explicitly as a subscript.

1.2. **Main result.**

We are interested in providing expressions for the Poincaré and logarithmic Sobolev constant of a ϱ-metastable Markov chain in the following sense.

Definition 1.7 (Poincaré and logarithmic Sobolev constant)**.** The Poincaré constant $C_{\mathrm{PI}} \equiv C_{\mathrm{PI}}(P,\mu)$ is defined by

$$C_{\mathrm{PI}} \;:=\; \sup\bigl\{\mathrm{Var}_\mu[f] \,:\, f \in \ell^2(\mu) \text{ such that } \mathscr{E}(f) = 1\bigr\}, \tag{1.11}$$

whereas the logarithmic Sobolev constant $C_{\mathrm{LSI}} \equiv C_{\mathrm{LSI}}(P,\mu)$ is given by

$$C_{\mathrm{LSI}} \;:=\; \sup\bigl\{\mathrm{Ent}_\mu[f^2] \,:\, f \in \ell^2(\mu) \text{ such that } \mathscr{E}(f) = 1\bigr\}. \tag{1.12}$$

Let $\{F_i : i \in I\} = \mathscr{M} \cup \{\{x\} : x \in \mathscr{S} \setminus \bigcup_{i=1}^K M_i\}$ be a partition of $\mathscr{S}$ and denote by $\mathscr{F} := \sigma(F_i : i \in I)$ the corresponding σ-algebra. Further, for any $f \in \ell^2(\mu)$ define the conditional expectation $\mathrm{E}_\mu[f \,|\, \mathscr{F}] \colon \mathscr{S} \to \mathbb{R}$ by

$$\mathrm{E}_\mu[f \,|\, \mathscr{F}](x) := \mathrm{E}_\mu[f \,|\, F_i] \qquad \Longleftrightarrow \qquad F_i \ni x.$$

The starting point for proving sharp estimates of both the Poincaré and the logarithmic Sobolev constant in the context of metastable Markov chains is a splitting of the variance and the entropy into the contribution within and outside the metastable sets. By using the projection property of the conditional expectation, we obtain for any $f \in \ell^2(\mu)$

$$\mathrm{Var}_\mu[f] = \sum_{i=1}^K \mu[M_i] \, \mathrm{Var}_{\mu_{M_i}}[f] + \mathrm{Var}_\mu\big[\mathrm{E}_\mu[f \,|\, \mathscr{F}]\big] \tag{1.13}$$

$$\mathrm{Ent}_\mu[f^2] = \sum_{i=1}^K \mu[M_i] \, \mathrm{Ent}_{\mu_{M_i}}[f^2] + \mathrm{Ent}_\mu\big[\mathrm{E}_\mu[f^2 \,|\, \mathscr{F}]\big] \tag{1.14}$$

Our main result relies on the following regularity condition and an assumption on the Poincaré and logarithmic Sobolev constants within the metastable sets.

Assumption 1.8 (Regularity condition)**.** *Assume that there exists* $\eta \in [0, 1)$ *such that*

$$\mathrm{Var}_{\mu_{M_i}}\left[\frac{\nu_{M_i, M_j}}{\mu_{M_i}}\right] \le \frac{\eta \, \mu[M_i]}{\mathrm{cap}(M_i, M_j)} \qquad \forall\, M_i, M_j \in \mathscr{M} \quad \textit{with } i \neq j.$$

Remark 1.9. Since $e_{M_i, M_j}(x) \le \mathrm{cap}(M_i, M_j)/\mu(x)$ for all $x \in M_i$, the following trivial upper bound on η holds

$$\eta \le \max\Big\{1, \varrho \, |M_i| \max_{x, y \in M_i} \{\mu(x)/\mu(y)\}\Big\}.$$

Hence, $\eta \ll 1$ if for each metastable set M_i both its cardinality and the fluctuations of the invariant distribution μ on it are sufficiently small compared to ϱ.

Assumption 1.10. *Assume that for any* $i \in \{1, \dots, K\}$

(i) $$C_{\mathrm{PI}, i} = \sup\Big\{ \mathrm{Var}_{\mu_{M_i}}[f] \;:\; f \in \ell^2(\mu) \textit{ such that } \mathscr{E}(f) = 1\Big\} < \infty, \tag{1.15}$$

(ii) $$C_{\mathrm{LSI}, i} = \sup\Big\{ \mathrm{Ent}_{\mu_{M_i}}[f^2] \;:\; f \in \ell^2(\mu) \textit{ such that } \mathscr{E}(f) = 1\Big\} < \infty. \tag{1.16}$$

We set

$$C_{\mathrm{PI}, \mathscr{M}} := \max\Big\{1, \sum_{i=1}^K \mu[M_i] \, C_{\mathrm{PI}, i}\Big\} \qquad \text{and} \qquad C_{\mathrm{LSI}, \mathscr{M}} := \max\Big\{1, \sum_{i=1}^K \mu[M_i] \, C_{\mathrm{LSI}, i}\Big\}.$$

Remark 1.11. Note that for any $M \in \mathscr{M}$ that consists of a single point, i.e. $|M| = 1$, Assumptions 1.8 and 1.10 are satisfied for $\eta = 0$ and $C_{\mathrm{PI}, \mathscr{M}} = C_{\mathrm{LSI}, \mathscr{M}} = 1$.

The first result concerns the mean hitting times between metastable states. We obtain an asymptotically expression in terms of capacities, if we in addition assume a bound on the asymmetry of the involved local minima.

Proposition 1.12. *Suppose that Assumption 1.3 holds with $K \geq 2$. For $M_i \in \mathscr{M}$ define*

$$\mathscr{M}_i \;:=\; \{M \in \mathscr{M} \setminus \{M_i\} \,:\, \mu[M] \geq \mu[M_i]\}.$$

Let $J \equiv J(i) \subset \{1,\dots,K\} \setminus \{i\}$ the corresponding index set, that is $\mathscr{M}_i = \{M_j : j \in J\}$. Assume that $B = \bigcup_{j\in J} M_j \neq \emptyset$ and that there exists $\delta \in [0,1)$ such that $\mu[\mathscr{S}_j] \leq \delta \mu[\mathscr{S}_i]$ for all $j \notin J \cup \{i\}$. Additionally, assume that

$$C_{\mathrm{ratio}} \;:=\; \max_{j\in J} \mu[\mathscr{S}_j]/\mu[\mathscr{S}_i] \;<\; \infty$$

is such that $\varrho \ln C_{\mathrm{ratio}/\varrho} \ll 1$. Then,

$$\mathbb{E}_{\nu_{M_i,B}}[\tau_B] \;=\; \frac{\mu[\mathscr{S}_i]}{\operatorname{cap}(M_i,B)}\Big(1 + O\big(\delta + \varrho \ln(C_{\mathrm{ratio}}/\varrho)\big)\Big).$$

Let us state for the sake of presentation the main result in the case of two metastable sets $K = 2$. For the statement in the case of $K > 2$, we refer to Theorem 3.5 and Theorem 3.9.

Theorem 1.13. *Suppose that the Assumptions 1.3 with $K = 2$, 1.8 and 1.10 i) hold such that $C_{\mathrm{PI},\mathscr{M}}(\varrho + \eta) \ll 1$. Then, it holds that*

$$C_{\mathrm{PI}} \;=\; \frac{\mu[\mathscr{S}_1]\,\mu[\mathscr{S}_2]}{\operatorname{cap}(M_1,M_2)}\Big(1 + O\Big(\sqrt{C_{\mathrm{PI},\mathscr{M}}(\varrho+\eta)}\Big)\Big). \tag{1.17}$$

Moreover, if in addition Assumption 1.10 ii) holds and

$$C_{\mathrm{mass}} \;:=\; \max_{i\in\{1,\dots,K\}} \max_{x\in\mathscr{S}_i} \ln\big(1 + \mathrm{e}^2/\mu_i(x)\big) \;<\; \infty \tag{1.18}$$

such that $C_{\mathrm{mass}}\, C_{\mathrm{LSI},\mathscr{M}}(\varrho+\eta) \ll 1$. Then, it holds that

$$C_{\mathrm{LSI}} \;=\; \frac{1}{\Lambda(\mu[\mathscr{S}_1],\mu[\mathscr{S}_2])}\,\frac{\mu[\mathscr{S}_1]\,\mu[\mathscr{S}_2]}{\operatorname{cap}(M_1,M_2)}\Big(1 + O\Big(\sqrt{C_{\mathrm{mass}}\, C_{\mathrm{LSI},\mathscr{M}}(\varrho+\eta)}\Big)\Big), \tag{1.19}$$

where $\Lambda(\alpha,\beta) := \int_0^1 \alpha^s \beta^{1-s}\,\mathrm{d}s = (\alpha-\beta)/\ln\frac{\alpha}{\beta}$ for $\alpha,\beta > 0$ is the logarithmic mean.

Remark 1.14. By using a standard linearization argument in (1.12), it follows that $C_{\mathrm{LSI}} \geq 2C_{\mathrm{PI}}$, see [9, Proposition 5]. Notice that in the symmetric case when $\mu[\mathscr{S}_1]\big(1+o(1)\big) = \frac{1}{2} = \mu[\mathscr{S}_2]\big(1+o(1)\big)$ we have that $C_{\mathrm{LSI}} = 2C_{\mathrm{PI}}\big(1+o(1)\big)$. Let us note that Assumption (1.18) restricts the result on the logarithmic Sobolev constant to finite state spaces.

Corollary 1.15. *Suppose that the assumptions of Theorem 1.13 hold. Further, assume that $C_{\mathrm{ratio}}^{-1} \ll 1$ and $\varrho \ln C_{\mathrm{ratio}/\varrho} \ll 1$. Then,*

$$C_{\mathrm{PI}} \;=\; \mathbb{E}_{\nu_{M_2,M_1}}\big[\tau_{M_1}\big]\Big(1 + O\Big(C_{\mathrm{ratio}}^{-1} + \varrho \ln(C_{\mathrm{ratio}}/\varrho) + \sqrt{C_{\mathrm{PI},\mathscr{M}}(\varrho+\eta)}\Big)\Big).$$

Let us comment on similar results in the literature.

First, the quantity on the right hand side of (1.17) bears some similarity to the Cheeger constant [19] on weighted graphs [27] defined by

$$C_{\mathrm{Cheeger}} \;:=\; \sup_{A\subset\mathscr{S}:\mu[A]\in(0,1)} \frac{\mu[A]\,\mu[A^c]}{\operatorname{cap}(A,A^c)}$$

with $A^c := \mathcal{S} \setminus A$. Moreover, we note that $\operatorname{cap}(A, A^c) = -\langle \mathbb{1}_A, L\mathbb{1}_{A^c}\rangle_\mu$. Then, the main result of [27, Theorem 2.1] translated to the current setting reads

$$C_{\text{Cheeger}} \leq C_{\text{PI}} \leq 8C_{\text{Cheeger}}^2.$$

Hence, the main result (1.17) can be seen as an asymptotic sharp version of the Cheeger estimate in the metastable setting.

Let us point out that a different approach to the construction of suitable local equilibrium states is based on quasi-stationary distributions [7]. If the state space is decomposed into a partition $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, then there are two canonical restrictions of the dynamic to the elements of the partition by either imposing absorbing (Dirichlet) or reflecting (Neumann) boundary conditions. Metastability is characterized in terms of ratios between Dirichlet and Neumann spectral gaps of the restricted generator. This provides explicit estimates on the relaxation rates toward the quasi-stationary distribution inside of each partition. They obtain in [7, Theorem 2.10] a result bearing some similarity to (1.17), where the capacity $\operatorname{cap}(M_1, M_2)$ is replaced by so called (κ, λ)-capacities between $\overline{M}_1$ and $\overline{M}_2$. These capacities are obtained by extending the state space by copies of the sets M_1 and M_2 and attaching those with conductivity κ and λ to the original elements of the sets. The error bound in this formulation depend on a careful choice of κ and λ in terms the metastability ratios.

1.3. **Random field Curie-Weiss model.** One particular class of models, we are interested in, are *disordered mean field spin systems*. As an example, we consider the Ising model on a complete graph, say on $N \in \mathbb{N}$ vertices, also known as Curie-Weiss model, in a random magnetic field. The state space of this model is $\mathcal{S} = \{-1, 1\}^N$. The *random Hamiltonian* is given by

$$H(\sigma) := -\frac{1}{2N} \sum_{i,j=1}^{N} \sigma_i \sigma_j - \sum_{i=1}^{N} h_i \sigma_i, \qquad \sigma \in \mathcal{S}, \tag{1.20}$$

where $h \equiv (h_i : i \in \{1, \ldots, N\})$ is assumed to be a family of i.i.d. random variable on $\mathbb{R}$ distributed according to $\mathbb{P}^h$ with bounded support, that is

$$\exists\, h_\infty \in (0, \infty): \qquad |h_i| \leq h_\infty \qquad \mathbb{P}^h\text{-almost surely.} \tag{1.21}$$

The *random Gibbs measure* on $\mathcal{S}$ is defined by

$$\mu(\sigma) := Z^{-1} \exp(-\beta H(\sigma))\, 2^{-N},$$

where $\beta \geq 0$ is the inverse temperature and Z is the normalization constant also called *partition function*. The *Glauber dynamics*, that we consider, is a Markov chain $(\sigma(t) : t \geq 0)$ in discrete-time with *random transition probabilities*

$$p(\sigma, \sigma') := \frac{1}{N} \exp\bigl(-\beta [H(\sigma') - H(\sigma)]_+\bigr)\, \mathbb{1}_{|\sigma - \sigma'|_1 = 2}, \tag{1.22}$$

where $[x]_+ := \max\{x, 0\}$ and $p(\sigma, \sigma) = 1 - \sum_{\sigma' \in \mathcal{S}} p(\sigma, \sigma')$. Notice that, for each realization of the magnetic field h, the Markov chain is ergodic and reversible with respect to the Gibbs measure μ.

Various stationary and dynamic aspects of the random field Curie-Weiss model has been studied. In particular, the metastable behaviour of this model has been analyzed in great detail in [12, 5, 6], where the potential theoretic approach was used to compute precisely metastable exit times and to prove the asymptotic exponential distribution of normalized metastable exit times. For an excellent review we refer to the recent monograph [11, Chapter 14 and 15]. Estimates on the spectral gap has been derived in [32] in the particular simple cases where the random field takes only two values $\pm\varepsilon$ and the parameters are chosen in such a way that only two minima are present.

A particular feature of this model is that it allows to introduce *mesoscopic variables* by using a suitable coarse-graining procedure such that the induced dynamics is well approximated by a Markov process. Let $I^h := [-h_\infty, h_\infty]$ denote the support of $\mathbb{P}^h$. For any $n \in \mathbb{N}$ we find a partition of I^h such that $|I^h_\ell| \leq 2h_\infty/n$ and $I^h = \bigcup_{\ell=1}^n I^h_\ell$. Hence, each realization of h induces a partition of the set $\{1,\dots,N\}$ into mutually disjoint subsets

$$\Lambda_\ell := \big\{ i \in \{1,\dots,N\} : h_i \in I^h_\ell \big\}, \qquad \ell \in \{1,\dots,n\}.$$

Based on this partition, consider the mesoscopic variable $\rho : \mathscr{S} \to \Gamma^n \subset [-1,1]^n$,

$$\rho(\sigma) = \big(\rho_1(\sigma),\dots,\rho_n(\sigma)\big) \qquad \text{with} \qquad \rho_\ell(\sigma) := \frac{1}{N}\sum_{i\in\Lambda_\ell}\sigma_i, \quad \ell \in \{1,\dots,n\},$$

that serves as an n-dimensional *order parameter*. A crucial feature of the mean field model is that the Hamiltonian (1.20) can be rewritten as a function of the mesoscopic variable. In order to do so, for any $\ell \in \{1,\dots,n\}$ the block-averaged field and its fluctuations are defined by

$$\bar{h}_\ell := \frac{1}{|\Lambda_\ell|}\sum_{i\in\Lambda_\ell} h_i \qquad \text{and} \qquad \tilde{h}_i := h_i - \bar{h}_\ell, \quad \forall\, i \in \Lambda_\ell.$$

Then,

$$H(\sigma) = -N E(\rho(\sigma)) - \sum_{\ell=1}^n \sum_{i\in\Lambda_\ell} \sigma_i \tilde{h}_i,$$

where the function $E : \mathbb{R}^n \to \mathbb{R}$ is given by $E(\boldsymbol{x}) = \frac{1}{2}\big(\sum_{l=1}^n \boldsymbol{x}_\ell\big)^2 + \sum_{\ell=1}^n \bar{h}_\ell \boldsymbol{x}_\ell$. We define the distribution of ρ under the Gibbs measure as the *induced measure*

$$\boldsymbol{\mu}(\boldsymbol{x}) := \mu \circ \rho^{-1}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Gamma^n.$$

Further, we introduce the *mesoscopic free energy* $F : \mathbb{R}^n \to \mathbb{R}$ which is defined by

$$F(\boldsymbol{x}) := E(\boldsymbol{x}) + \frac{1}{\beta}\sum_{\ell=1}^n \frac{|\Lambda_\ell|}{N} I_\ell(N\boldsymbol{x}_\ell/|\Lambda_\ell|), \tag{1.23}$$

where for any $\ell \in \{1,\dots,n\}$ the *entropy* I_ℓ is given as the Legendre-Fenchel dual of

$$t \mapsto \frac{1}{N}\sum_{i\in\Lambda_\ell} \ln\cosh(t + \beta\tilde{h}_i).$$

Notice that the distribution $\boldsymbol{\mu}$ satisfies a sharp large deviation principle with scale N and rate function F. The structure of the mesoscopic free energy landscape has been analyzed in great detail in [5]. In particular, $\boldsymbol{z} \in \mathbb{R}^n$ is a critical point of F, if and only if, for all $\ell \in \{1,\dots,n\}$

$$\boldsymbol{z}_\ell \;=\; \frac{1}{N} \sum_{i \in \Lambda_\ell} \tanh(\beta(z(\boldsymbol{z}) + h_i))$$

where $z(\boldsymbol{z}) = \sum_{\ell=1}^n \boldsymbol{z}_\ell \in \mathbb{R}$ solves in addition the equation $z = \frac{1}{N}\sum_{i=1}^N \tanh(\beta(z+h_i))$. It turns out that $\boldsymbol{z}$ is a critical point of index 1, if $\frac{\beta}{N}\sum_{i=1}^N 1 - \tanh^2(\beta(z(\boldsymbol{z})+h_i)) > 1$. Moreover, at any critical point $\boldsymbol{z}$ the value of the mesoscopic free energy can be computed explicitly and is given by $F(\boldsymbol{z}) = \frac{1}{2}z(\boldsymbol{z})^2 - \frac{1}{\beta N}\sum_{i=1}^N \ln\cosh(\beta(z(\boldsymbol{z})+h_i))$. Let us stress out the fact that the topology of the mesoscopic energy landscape is independent of the parameter n.

Assumption 1.16. *Assume that distribution $\mathbb{P}^h$ of the random field is such that, $\mathbb{P}^h$-a.s., there exist $\beta > 0$ and $N_0(h) < \infty$ such that for all $N \geq N_0(h)$ and $n \geq 1$ the mesoscopic free energy $F: \Gamma^n \to \mathbb{R}$ has $K \geq 2$ local minima.*

Remark 1.17. Without the external field, i.e. $h_i \equiv 0$ for all i, the free energy has two local minima if $\beta > 1$.

Denote by $\boldsymbol{m}_i \in \Gamma^n$, $i \in \{1,\dots,K\}$, the best lattice approximation of the corresponding local minima. We choose the label of $\boldsymbol{m}_i$ by the following procedure: First, define for any non-empty, disjoint $\boldsymbol{A},\boldsymbol{B} \subset \Gamma^n$ the *communication height*, $\Phi(\boldsymbol{A},\boldsymbol{B})$, between $\boldsymbol{A}$ and $\boldsymbol{B}$ by

$$\Phi(\boldsymbol{A},\boldsymbol{B}) \;=\; \min_{\gamma} \max_{\boldsymbol{x}\in\gamma} F(\boldsymbol{x}), \tag{1.24}$$

where the minimum is taken over all nearest-neighbour path in Γ^n that connects $\boldsymbol{A}$ and $\boldsymbol{B}$. Then, the label are chosen in such a way that, with $\boldsymbol{M}_k := \{\boldsymbol{m}_1,\dots,\boldsymbol{m}_k\}$,

$$\Delta_{k-1} \;:=\; \Phi(\boldsymbol{m}_k, \boldsymbol{M}_{k-1}) - F(\boldsymbol{m}_k) \;\leq\; \min_{i<k}\{\Phi(\boldsymbol{m}_i, \boldsymbol{M}_k \setminus \boldsymbol{m}_i) - F(\boldsymbol{m}_i)\}$$

for all $k = K,\dots,2$. Notice that, by construction, $\Delta_1 \geq \Delta_2 \geq \ldots \geq \Delta_{K-1} > 0$.

To obtain matching upper and lower bounds in the application of Theorem 1.13 to the random field Curie-Weiss model in case $K \geq 3$, we impose the following non-degeneracy condition on the largest communication height.

Assumption 1.18 (Non-degeneracy condition)**.** *For $K \geq 3$ assume that there exists $\theta > 0$ and $N_1(h) < \infty$ such that, $\mathbb{P}^h$-a.s.,*

$$\Delta_1 - \Delta_2 \;\geq\; \theta, \qquad \forall\, N \geq N_1(h).$$

Remark 1.19. Note that under the non-degeneracy condition the mesoscopic free energy landscape may still have more than one global minima.

In the sequel, we first impose conditions on the finiteness of the coarse-graining controlled by the parameter n. Depending on the choice of n the state space dimension N has to be larger then a certain threshold. In this sense, the results hold by first letting $N \to \infty$ and then $n \to \infty$.

Proposition 1.20 (ϱ-metastability)**.** *Suppose that Assumption 1.16 holds. For $K = 2$ set $\theta = \Delta_1$ and $N_1(h) = 1$. If $K \geq 3$ assume additionally that Assumption 1.18 is satisfied. Then, for any $c_1 \in (0, \theta)$ there exists $n_0 \equiv n_0(\theta, c_1)$ such that for all $n \geq n_0$ there exists $N_2 < \infty$ such that for all $N \geq N_0(h) \vee N_1(h) \vee N_2$ the random field Curie-Weiss model is $\varrho := \mathrm{e}^{-c_1 \beta N}$-metastable with respect to $\mathscr{M} := \{M_1, M_2\}$, where $M_1 := \rho^{-1}(\boldsymbol{m}_1)$ and $M_2 := \rho^{-1}(\boldsymbol{m}_2)$.*

Our main result in this subsection is the following.

Theorem 1.21. *Suppose the assumptions of Proposition 1.20 hold with $\varrho = \mathrm{e}^{-c_1 \beta N}$. Then, for any $c_2 \in (0, c_1/2)$, there exists $n_1 \equiv n_1(c_1, c_2, \beta, h_\infty) < \infty$ such that for any $n \geq n_0 \vee n_1$ there exists $N_2 < \infty$ such that for all $N \geq N_0(h) \vee N_1(h) \vee N_2$ the random field Curie-Weiss model satisfies a Poincaré inequality with constant*

$$C_{\mathrm{PI}} = \frac{\mu[\mathscr{S}_1]\,\mu[\mathscr{S}_2]}{\mathrm{cap}(M_1, M_2)} \Big(1 + O\big(\mathrm{e}^{-c_2 \beta N}\big)\Big) \tag{1.25}$$

as well as a logarithmic Sobolev inequality with constant

$$C_{\mathrm{LSI}} = \frac{C_{\mathrm{PI}}}{\Lambda(\mu[\mathscr{S}_1], \mu[\mathscr{S}_2])} \Big(1 + O\big(\mathrm{e}^{-c_2 \beta N}\big)\Big). \tag{1.26}$$

Let us highlight that this result is valid in the symmetric ($F(\boldsymbol{m}_1) = F(\boldsymbol{m}_2)$) as well asymmetric case ($F(\boldsymbol{m}_1) \neq F(\boldsymbol{m}_2)$). Moreover, the capacities between pairs of metastable sets are calculated asymptotically with explicit error bounds in [12, 5, 6, 11]. Hence, the right hand side of (1.25) and (1.26) can be made asymptotically explicit in terms of the free energy (1.23).

Corollary 1.22. *Suppose that the assumptions of Theorem 1.21 hold with $\varrho = \mathrm{e}^{-c_1 \beta N}$. Then, for any $c_2 \in (0, \min\{c_1/2, F(\boldsymbol{m}_2) - F(\boldsymbol{m}_1)\})$ it holds*

$$C_{\mathrm{PI}} = \mathbb{E}_\sigma\big[\tau_{M_1}\big] \Big(1 + O\big(\mathrm{e}^{-c_2 \beta N}\big)\Big), \qquad \forall\, \sigma \in M_2. \tag{1.27}$$

Proof. In view of [5, Equation (3.16)], $C_{\mathrm{ratio}}^{-1} := \mu[\mathscr{S}_2]/\mu[\mathscr{S}_1] = O(\mathrm{e}^{-\beta \delta N})$ for any $\delta \in (0, F(\boldsymbol{m}_2) - F(\boldsymbol{m}_1))$. Thus, (1.27) immediate consequence of Corollary 1.15 and [6, Theorem 1.1] □

Finally, notice that sharp asymptotics of the mean hitting time including the precise prefactor has been establish in [5].

2. Functional inequalities

The results in this section consider functional inequalities, which do not make any explicit reference to time. Therefore, the results hold in the more general setting of L as defined in (1.1) being the generator of a continuous time Markov chain on a countable state space $\mathscr{S}$. This accounts to dropping the normalization condition $\sum_y p(x,y) = 1$ and assuming $p(x,y)$ to be the elements of the infinitesimal generator satisfying

$$\forall\, x \neq y : p(x,y) \geq 0, \quad \forall x : 0 \leq -p(x,x) < \infty \quad \text{and} \quad \sum_{y\in\mathscr{S}} p(x,y) = 0.$$

We refer to [11, Chapter 7.2.2] for the general relation of hitting times between discrete time and continuous time Markov chains.

2.1. **Capacitary inequality.** The capacitary inequality is a generalization of the co-area formula. For Sobolev functions on $\mathbb{R}^d$ it has been first proven by Maz'ya in [33]. For a comprehensive treatment with further applications we refer to the books of Maz'ya [34], Chen [20] and Bakry, Gentil and Ledoux [2].

Theorem 2.1 (Capacitary inequality). *For any $f \in \ell^2(\mu)$ and any $t \in [0,\infty)$, let A_t be the super level-set of f, that is*

$$A_t := \{x \in \mathscr{S} \,:\, |f(x)| > t\}. \tag{2.1}$$

Let $B \subset \mathscr{S}$ be non-empty, then for any function $f : \mathscr{S} \to \mathbb{R}$ with $f|_B \equiv 0$ it holds that

$$\int_0^\infty 2t \,\operatorname{cap}(A_t, B)\,\mathrm{d}t \;\leq\; 4\,\mathscr{E}(f). \tag{2.2}$$

Proof. Due to the fact that $\mathscr{E}(|f|) \leq \mathscr{E}(f)$, let us assume without lost of generality that $f(x) \geq 0$ for all $x \in \mathscr{S}$. To lighten notation, for any $t \in [0,\infty)$, we denote $h_t := h_{A_t,B}$ the equilibrium potential as defined in (1.4). Since $\operatorname{supp} Lh_t = A_t \cup B$, $f|_B \equiv 0$ and $f|_{A_t} > t$, it follows

$$t\,\operatorname{cap}(A_t,B) \;\leq\; \langle -Lh_t, f\rangle_\mu \;=\; \frac{1}{2}\sum_{x,y\in\mathscr{S}} \mu(x)\,p(x,y)\big(f(x)-f(y)\big)\big(h_t(x)-h_t(y)\big).$$

An application of the Cauchy-Schwarz inequality yields

$$\begin{aligned}&\int_0^\infty 2t\,\operatorname{cap}(A_t,B)\,\mathrm{d}t\\ &\qquad\leq\; 2\,\mathscr{E}(f)^{1/2}\bigg(\frac{1}{2}\sum_{x,y\in\mathscr{S}}\mu(x)\,p(x,y)\bigg(\int_0^\infty \big(h_t(x)-h_t(y)\big)\,\mathrm{d}t\bigg)^2\bigg)^{1/2}.\end{aligned}$$

Now, we use the following identity: for any function $g \in L^1([0,\infty))$ holds

$$\bigg(\int_0^\infty g(t)\,\mathrm{d}t\bigg)^2 = \int_0^\infty\int_0^\infty g(t)\,g(s)\,\mathrm{d}s\,\mathrm{d}t = 2\int_0^\infty\int_0^t g(t)\,g(s)\,\mathrm{d}s\,\mathrm{d}t.$$

This allows to rewrite

$$\int_0^\infty 2t\,\operatorname{cap}(A_t,B)\,\mathrm{d}t \;\leq\; 2\,\mathscr{E}(f)^{1/2}\bigg(2\int_0^\infty\int_0^t \langle -Lh_t, h_s\rangle_\mu\,\mathrm{d}s\,\mathrm{d}t\bigg)^{1/2}.$$

Finally, by exploiting additionally the fact that $A_t \subset A_s$ for all $t \geq s$, we obtain that $\langle -Lh_t, h_s\rangle_\mu = \operatorname{cap}(A_t,B)$ and the assertion of the theorem follows. □

2.2. **Orlicz-Birnbaum estimates.** Let us assume for a moment that for some constant $C_{\mathrm{CI}} > 0$ a measure-capacity comparison for the form $C_{\mathrm{CI}} \operatorname{cap}(A,B) \geq \mu[A]$ for all $A \subset \mathscr{S} \setminus B$ is valid. Then a combination of the capacitary inequality (2.2) with

$$\mathrm{E}_\mu\big[f^2\big] = \int_0^\infty 2t\,\mu[A_t]\,\mathrm{d}t,$$

leads to $\mathrm{E}_\mu[f^2] \leq 4C_{\mathrm{CI}}\,\mathscr{E}(f)$ for all f with $f|_B \equiv 0$. This observation due to [33] provides estimates on the Dirichlet eigenvalue of the generator L.

This strategy can be generalized to the ℓ^p case and more generally to logarithmic Sobolev constants by introducing suitable Orlicz spaces. We provide Poincaré inequalities with respect to Orlicz-norms under suitable measure-capacity inequalities. For diffusion processes on $\mathbb{R}^d$ similar results are contained in [2, Chapter 8].

Definition 2.2 (Orlicz space [39, Section 1.3])**.** A function $\Phi\colon [0,\infty) \to [0,\infty]$ is a *Young function* if it is convex, $\Phi(0) = \lim_{r\to 0}\Phi(r) = 0$ and $\lim_{r\to\infty}\Phi(r) = \infty$. Then, its *Legendre-Fenchel dual* $\Psi\colon [0,\infty) \to [0,\infty]$ defined by

$$\Psi(r) = \sup_{s\in[0,\infty]}\big\{sr - \Phi(s)\big\}$$

is also a Young function (cf. Lemma A.1) and two functions with these properties are called *Legendre-Fenchel* pair. For some $K > 0$ the Orlicz-norm for functions $f \in \ell^1(\mu)$ is defined by

$$\|f\|_{\Phi,\mu,K} := \sup\Big\{\mathrm{E}_\mu\big[|f|g\big] : g \geq 0,\ \mathrm{E}_\mu\big[\Psi(g)\big] \leq K\Big\}, \tag{2.3}$$

and $\|f\|_{\Phi,\mu} := \|f\|_{\Phi,\mu,1}$ to lighten notation. The space of *Orlicz functions*, $\ell^\Phi(\mu,K) \subset \ell^1(\mu)$, is the set of summable functions f on $\mathscr{S}$ with finite Orlicz-norm.

Lemma 2.3. *For any $A \subset \mathscr{S}$ holds*

$$\|\mathbb{1}_A\|_{\Phi,\mu,K} = \mu[A]\,\Psi^{-1}\bigg(\frac{K}{\mu(A)}\bigg), \tag{2.4}$$

where $\Psi^{-1}(t) := \inf\big\{s \in [0,\infty] : \Psi(s) > t\big\}$.

Proof. One direction follows from using the function $g(x) = \mathbb{1}_A(x)\,\Psi^{-1}(K/\mu[A])$ in the definition of (2.3). Using the fact that Ψ^{-1} is concave (cf. Lemma A.1), the estimate in the other direction follows from

$$\mathrm{E}_\mu\big[\mathbb{1}_A\,g\big] = \mu[A]\,\mathrm{E}_\mu\bigg[\frac{\mathbb{1}_A}{\mu[A]}\,(\Psi^{-1}\circ\Psi)(g)\bigg] \leq \mu[A]\,\Psi^{-1}\bigg(\frac{1}{\mu[A]}\,\mathrm{E}_\mu\big[\mathbb{1}_A\,\Psi(g)\big]\bigg).$$

Taking finally the supremum over all g with $\mathrm{E}_\mu[\Psi(g)] \leq K$ concludes the proof. □

Example 2.4. The following Legendre-Fenchel pairs are stated for later reference:

a) For $p \in (1,\infty)$: $(\Phi_p(r),\Psi_p(r)) := \big(\frac{1}{p}r^p, \frac{1}{p^*}r^{p^*}\big)$ with $1/p + 1/p^* = 1$ the resulting Orlicz norm is equivalent to the usual $\ell^p(\mu)$ spaces. The limiting pair

$p \to 1$ is given by $\Phi_1(r) = r$ and $\Psi_1 : [0,\infty) \to [0,\infty]$ by

$$\Psi_1(r) = \begin{cases} 0, & r \le 1 \\ \infty, & r > 1 \end{cases} \quad \text{and hence} \quad \Psi_1^{-1}(r) = \begin{cases} 0, & r = 0 \\ 1, & r > 0 \end{cases}.$$

b) $(\Phi_{\mathrm{Ent}}(r), \Psi_{\mathrm{Ent}}(r)) := \big(\mathbb{1}_{[1,\infty)}(r)(r\ln r - r + 1), \mathrm{e}^r - 1\big)$ leads to a norm, which can be compared with the relative entropy

$$\forall f : \mathscr{S} \to \mathbb{R}_+, \qquad \mathrm{Ent}_\mu[f] \le \|f\|_{\Phi_{\mathrm{Ent}},\mu}.$$

Indeed, we use use the variational characterization of the entropy

$$\begin{aligned}\mathrm{Ent}_\mu[f] &= \sup_g \big\{ \mathrm{E}_\mu\big[f\, g\big] : \mathrm{E}_\mu\big[\mathrm{e}^g\big] \le 1 \big\} \\ &= \sup_h \big\{ \mathrm{E}_\mu\big[f \ln\big(\mathrm{e}^h - 1\big)\big] : h \ge 0,\, \mathrm{E}_\mu\big[\mathrm{e}^h - 1\big] \le 1 \big\} \le \|f\|_{\Phi_{\mathrm{Ent}},\mu},\end{aligned}$$

where the last step follows from (2.3) by noting that $\ln(\mathrm{e}^h - 1) \le h$.

Proposition 2.5 (Orlicz-Birnbaum estimates)**.** *Let $B \subset \mathscr{S}$ and $\nu \in \mathscr{P}(\mathscr{S})$. Then, for any Legendre-Fenchel pair (Φ, Ψ) there exist constants $C_\Phi, C_\Psi > 0$ satisfying*

$$C_\Psi \le C_\Phi \le 4\, C_\Psi,$$

such that the following statements are equivalent:

(i) For all sets $A \subset \mathscr{S} \setminus B$ the measure-capacity inequality holds

$$\nu[A]\, \Psi^{-1}\Big(\frac{K}{\nu[A]}\Big) \le C_\Psi \operatorname{cap}(A,B). \tag{2.5}$$

(ii) For all $f : \mathscr{S} \to \mathbb{R}$ such that $f \in \ell^2(\mu)$ and $f|_B \equiv 0$, it holds that

$$\big\|f^2\big\|_{\Phi,\nu,K} \le C_\Phi\, \mathscr{E}(f). \tag{2.6}$$

Proof. (i) $\Rightarrow$ (ii): Let $G_{\Psi,K} := \{g : g \ge 0, \mathrm{E}_\nu[\Psi(g)] \le K\}$, take $f \in \ell^2(\mu)$ with finite support and let A_t be the super-level set of f as defined in (2.1). Then, we have that

$$\big\|f^2\big\|_{\Phi,\nu,K} \overset{(2.3)}{=} \sup_{g \in G_{\Psi,K}} \mathrm{E}_\nu\big[f^2 g\big] \le \int_0^\infty 2t \sup_{g \in G_{\Psi,K}} \mathrm{E}_\nu\big[g\, \mathbb{1}_{A_t}\big]\, \mathrm{d}t = \int_0^\infty 2\, \big\|\mathbb{1}_{A_t}\big\|_{\Phi,\nu,K}\, \mathrm{d}t.$$

Thus, an application of Lemma 2.3 and Theorem 2.1 yields

$$\big\|f^2\big\|_{\Phi,\nu,K} \overset{(2.4)}{=} \int_0^\infty 2\, \nu[A_t]\, \Psi^{-1}\Big(\frac{K}{\nu[A_t]}\Big)\, \mathrm{d}t \overset{(2.5)}{\le} C_\Psi \int_0^\infty 2 \operatorname{cap}(A_t, B)\, \mathrm{d}t \overset{(2.2)}{\le} 4\, C_\Psi\, \mathscr{E}(f).$$

The case $f \in \ell^2(\mu)$ follows from dominated convergence, since $\mathscr{E}(f) \le \|f\|^2_{\ell^2(\mu)}$.
(ii) $\Rightarrow$ (i): Since $\mathscr{E}(f) \le \|f\|^2_{\ell^2(\mu)}$, we get from (2.6), that $f^2 \in \ell^\Phi(\nu, K)$. Hence, for any $f \in \ell^2(\mu)$ with $f|_A \equiv 1$ and $f|_B \equiv 0$ it holds that

$$\nu[A]\, \Psi^{-1}\Big(\frac{K}{\nu[A]}\Big) \overset{(2.4)}{=} \|\mathbb{1}_A\|_{\Phi,\nu,K} \le \big\|f^2\big\|_{\Phi,\nu,K} \overset{(2.6)}{\le} C_\Phi\, \mathscr{E}(f),$$

which by the Dirichlet principle (1.9) leads to (2.5). $\square$

Remark 2.6. Let us note, that either estimate (2.5) or (2.6) of Proposition 2.5 implies $\nu \ll \mu$ on $\mathscr{S}\setminus B$ with bounded density. Indeed, for $x \in \mathscr{S}\setminus B$ choose the function $\mathscr{S} \ni y \mapsto \mathbb{1}_x(y)$ as a test function in the Dirichlet principle (1.9) and apply (2.5). The same estimate can be obtained from (2.6) by considering again $\mathscr{S} \ni y \mapsto \mathbb{1}_x(y)$ and using the representation (2.4). In both cases, we deduce for any $x \in \mathscr{S} \setminus B$

$$\nu(x) \;\leq\; \frac{C}{\Psi^{-1}\big(K/\nu(x)\big)} \;\leq\; \frac{C}{\Psi^{-1}(K)}\,\mu(x),$$

where we used the monotonicity of $[0,\infty) \ni r \mapsto \Psi^{-1}(r)$. Hereby, C is either C_Ψ or C_Φ. Hence, $\nu \ll \mu$ and therefore $\ell^1(\mu) \subseteq \ell^1(\nu)$.

Remark 2.7. The result of Proposition 2.5 is a generalization of the Muckenhoupt criterion [37] for weighted Hardy inequalities, which was translated to the discrete setting for $\mathscr{S} = \mathbb{N}_0$ in [36]. The statement is, that for any $\nu, \mu \in \mathscr{P}(\mathbb{N}_0)$ and any $f : \{-1\} \cup \mathbb{N}_0 \to \mathbb{R}$ with $f(0) = f(-1) = 0$ the inequality

$$\sum_{x\geq 0} \nu(x) f(x)^2 \;\leq\; C_1 \sum_{x\geq 0} \mu(x)\big(f(x+1) - f(x)\big)^2 \tag{2.7}$$

holds if and only if

$$C_2 \;=\; \sup_{x\geq 1}\left(\sum_{y=0}^{x-1} \frac{1}{\mu[y]}\right)\sum_{y\geq x} \nu[y] \;<\; \infty.$$

In this case the constants satisfy $C_2 \leq C_1 \leq 4C_2$. This results can be deduced from Proposition 2.5 by using the Orlicz-pair (Φ_1, Ψ_1) from Example 2.4 a) and setting $B = \{0\}$. Then (2.6) becomes (2.7) for the (continuous time) generator

$$(Lf)(x) \;=\; (f(x+1) - f(x)) + \frac{\mu(x-1)}{\mu(x)}(f(x-1) - f(x))$$

and therefore $C_{\Phi_1} = C_1$. Notice that the equilibrium potential and hence the capacity along a one-dimensional, cycle-free path can be calculated explicitly (see e.g. [11, Section 7.1.4]). In particular, for any $x \in \mathbb{N}$ the solution $h_{x,0} \equiv h_{\{x,\ldots,\infty\},\{0\}}$ of the boundary value problem (1.4) on $\mathbb{N}_0$ is given by

$$h_{x,0}(y) \;=\; \sum_{z=y}^{x-1} \frac{1}{\mu(z)} \Bigg/ \sum_{z=0}^{x-1} \frac{1}{\mu(z)} \qquad \text{and} \qquad \operatorname{cap}\big(\{x,\ldots,\infty\},\{0\}\big) \;=\; \left(\sum_{z=0}^{x-1} \frac{1}{\mu(z)}\right)^{-1}.$$

In view of (2.5), this verifies that $C_{\Psi_1} = C_2$. The weighted Hardy inequality was then used to derive Poincaré and logarithmic Sobolev inequalities (cf. [8, 36, 1]), which we will do in a similar way in the following two Corollaries.

2.3. **Poincaré and Sobolev inequalities.** In order to deduce Poincaré or logarithmic Sobolev inequalities, it is not possible to directly apply Proposition 2.5. The reason is that the Orlicz-Birnbaum estimate (2.6) is with respect to Dirichlet test functions vanishing on a certain set, whereas the Poincaré and logarithmic Sobolev inequalities consider Neumann test function having average zero. Therefore, a splitting technique can be used to translate the Orlicz-Birnbaum estimate to the Neumann case. See also [20, Chapter 4.4] for some background on this technique. The additional step is taken care in the following two corollaries.

Corollary 2.8 (Poincaré inequalities)**.** *Let $\nu \in \mathscr{P}(\mathscr{S})$ and $b \in \mathscr{S}$. Then, there exist $C_{\mathrm{Var}}, C_{\mathrm{PI}} > 0$ satisfying*

$$\nu(b)\, C_{\mathrm{Var}} \;\leq\; C_{\mathrm{PI}} \;\leq\; 4\, C_{\mathrm{Var}} \tag{2.8}$$

such that the following statements are equivalent:

(i) For all $A \subset \mathscr{S} \setminus \{b\}$ the inequality holds

$$\nu[A] \;\leq\; C_{\mathrm{Var}}\, \mathrm{cap}(A, b). \tag{2.9}$$

(ii) The mixed Poincaré inequality holds, that is

$$\mathrm{Var}_\nu[f] \;\leq\; C_{\mathrm{PI}}\, \mathscr{E}(f), \qquad \forall f \in \ell^2(\mu).$$

Proof. (i) $\Rightarrow$ (ii): Let (Φ_1, Ψ_1) as in Example 2.4 a) and recall that $\Psi_1^{-1}|_{(0,\infty)} \equiv 1$. Then, the measure-capacity inequality (2.5) coincides exactly with (2.9). Hence,

$$\mathrm{Var}_\nu\big[f\big] \;=\; \min_{a \in \mathbb{R}} \mathrm{E}_\nu\big[(f-a)^2\big] \;\leq\; \big\|(f - f(b))^2\big\|_{\Phi_1,\nu} \overset{(2.6)}{\leq} 4\, C_{\mathrm{Var}}\, \mathscr{E}(f).$$

(ii) $\Rightarrow$ (i): We start with deducing a lower bound for the variance. Let $0 \leq f \leq 1$ be given such that $f|_A \equiv 1$ and $f(b) = 0$, then

$$\mathrm{Var}_\nu[f] \;=\; \frac{1}{2} \sum_{x,y \in \mathscr{S}} \nu(x)\, \nu(y)\, \big(f(x) - f(y)\big)^2 \;\geq\; \sum_{x \in A} \nu(x)\, \nu(b) \;=\; \nu[A]\, \nu(b).$$

The conclusion follows from the Dirichlet principle (1.9). □

Corollary 2.9 (Logarithmic Sobolev inequalities)**.** *Let $\nu \in \mathscr{P}(\mathscr{S})$ and $b \in \mathscr{S}$. Then, there exist $C_{\mathrm{Ent}}, C_{\mathrm{LSI}} > 0$ satisfying*

$$\frac{\nu(b)}{\ln(1 + \mathrm{e}^2)}\, C_{\mathrm{Ent}} \;\leq\; C_{\mathrm{LSI}} \;\leq\; 4\, C_{\mathrm{Ent}} \tag{2.10}$$

such that the following statements are equivalent:

(i) For all $A \subset \mathscr{S} \setminus \{b\}$ the inequality holds

$$\nu[A]\, \ln\!\left(1 + \frac{\mathrm{e}^2}{\nu[A]}\right) \;\leq\; C_{\mathrm{Ent}}\, \mathrm{cap}(A, b). \tag{2.11}$$

(ii) The mixed logarithmic Sobolev inequality holds, that is

$$\mathrm{Ent}_\nu[f^2] \;\leq\; C_{\mathrm{LSI}}\, \mathscr{E}(f), \qquad \forall f \in \ell^2(\mu). \tag{2.12}$$

Proof. (i) $\Rightarrow$ (ii): Set $f_b(x) := f(x) - f(b)$ for $x \in \mathcal{S}$. Then, by applying a useful observation due to Rothaus [2, Lemma 5.1.4],

$$\begin{aligned}\mathrm{Ent}_\nu\big[f^2\big] &\le \mathrm{Ent}_\nu\big[f_b^2\big] + 2\,\mathrm{E}_\nu\big[f_b^2\big]\\ &= \sup_g \big\{\mathrm{E}_\nu\big[f_b^2\,(g+2)\big] : \mathrm{E}_\nu\big[\mathrm{e}^g\big] \le 1\big\}\\ &\le \sup_h \big\{\mathrm{E}_\nu\big[f_b^2\,h\big] : h \ge 0,\ \mathrm{E}_\nu\big[\mathrm{e}^h - 1\big] \le \mathrm{e}^2\big\} \;=\; \big\|f_b^2\big\|_{\Phi,\nu,\mathrm{e}^2},\end{aligned} \tag{2.13}$$

where we used the Orlicz-Pair 2.4 b) and the definition of the K-Orlicz norm with $K = \mathrm{e}^2$ in (2.3). The first implication follows now by an application of (2.6).

(ii) $\Rightarrow$ (i): In order to prove the opposite direction, let $A \subset \mathcal{S} \setminus \{b\}$ with $\nu[A] \neq 0$, and consider a function $f : \mathcal{S} \to [0,1]$ with the property that $f|_A \equiv 1$ and $f(b) = 0$. By using $g = \ln(1/\nu[A])$ as test function in the variational representation of the entropy we deduce that

$$\mathrm{Ent}_\nu\big[f^2\big] \;\ge\; \sup_g \big\{\mathrm{E}_\nu\big[g\mathbb{1}_A\big] : \mathrm{E}_\nu[\mathrm{e}^g\mathbb{1}_A] \le 1\big\} \;\ge\; \nu[A]\,\ln\!\Big(\frac{1}{\nu[A]}\Big).$$

Since $\ln(1/x)/\ln(1+\mathrm{e}^2/x) \ge (1-x)/\ln(1+\mathrm{e}^2)$ for all $x \in (0,1]$ and $\nu[A] \in (0, 1-\nu(b)]$, we obtain that

$$\nu[A]\,\ln\!\Big(\frac{1}{\nu[A]}\Big) \;\ge\; \nu[A]\,\ln\!\Big(1 + \frac{\mathrm{e}^2}{\nu[A]}\Big)\,\frac{\nu(b)}{\ln(1+\mathrm{e}^2)}.$$

Thus, (2.11) follows from (2.12) by the Dirichlet principle (1.9). $\square$

The results of Corollary 2.8 and Corollary 2.9 can be strengthened to identify the optimal Poincaré and logarithmic Sobolev constant up to a universal numerical factor, i.e. replacing $\nu[b]$ in the lower bounds (2.8) and (2.10) by a universal numerical constant. The price to pay is to enforce the assumptions in the inequalities (2.9) and (2.11). Although, in the application to metastable Markov chains, these result cannot provide an asymptotic sharp constant, we include them here fore completeness.

Corollary 2.10. *Let $\nu \in \mathcal{P}(\mathcal{S})$. Then, there exist $C_{\mathrm{Var}}, C_{\mathrm{PI}} > 0$ satisfying*

$$\tfrac{1}{2} C_{\mathrm{Var}} \;\le\; C_{\mathrm{PI}} \;\le\; 4\,C_{\mathrm{Var}}$$

such that the following statements are equivalent:

(i) For all $A, B \subset \mathcal{S}$ disjoint with

$$\nu[A] \le \tfrac{1}{2} \quad \textit{and} \quad \nu[B] \ge \tfrac{1}{2} \qquad \textit{holds} \qquad \nu[A] \le C_{\mathrm{Var}}\,\mathrm{cap}(A,B). \tag{2.14}$$

(ii) The mixed Poincaré inequality holds, that is

$$\mathrm{Var}_\nu[f] \le C_{\mathrm{PI}}\,\mathcal{E}(f), \qquad \forall f \in \ell^2(\mu).$$

Proof. (i) $\Rightarrow$ (ii): Let $m \in \mathbb{R}$ denote the median of f with respect to ν, that is $\nu[f < m] \leq \frac{1}{2}$ and $\nu[f > m] \leq \frac{1}{2}$. Then, we can estimate

$$\mathrm{Var}_\nu[f] \leq \mathrm{E}_\nu\big[(f-m)_-^2\big] + \mathrm{E}_\nu\big[(f-m)_+^2\big] \leq 4C_{\mathrm{Var}}\Big(\mathscr{E}\big((f-m)_-\big) + \mathscr{E}\big((f-m)_+\big)\Big),$$

by an application of Proposition 2.5, which assumption is satisfied by (2.14) for the sets $A^- = \{f < m\}$ and $B^- = \{f \geq m\}$ for the first term and $A^+ = \{f > m\}$ and $B^+ = \{f \leq m\}$ for the second term. Then, the conclusion of the first implication follows by showing $\mathscr{E}((f-m)_+) + \mathscr{E}((f-m)_-) \leq \mathscr{E}(f)$. This estimate is a consequence of the pointwise bound for any $x, y \in \mathscr{S}$

$$\big((f(x)-m)_+ - (f(y)-m)_+\big)^2 + \big((f(x)-m)_- - (f(y)-m)_-\big)^2 \leq \big(f(x)-f(y)\big)^2.$$

Indeed, the bound is obvious for the cases $x, y \in \{f > m\}$ and $x, y \in \{f < m\}$. Now let $x \in \{f > m\}$ and $y \in \{f < m\}$, then the inequality reduces to show

$$(f(x)-m)^2 + (f(y)-m)^2 \leq (f(x)-f(y))^2,$$

which follows from the elementary inequality $mf(x) + mf(y) - m^2 \geq f(x)f(y)$ provided that $f(y) \leq m \leq f(x)$.

(ii) $\Rightarrow$ (i): For the converse statement let f be a test function such that $1 \geq f \geq \mathbb{1}_A$ and $f|_B \equiv 0$. Then, it follows that

$$\mathscr{E}(f)\, C_{\mathrm{PI}} \geq \mathrm{Var}_\mu[f] \geq \nu[B]\, \mathrm{E}_\nu[f^2] \geq \tfrac{1}{2}\, \nu[A].$$

Optimizing in f leads to the measure capacity inequality $\frac{1}{2}\nu[A] \leq C_{\mathrm{PI}} \operatorname{cap}(A, B)$. □

Corollary 2.11. *Let $\nu \in \mathscr{P}(\mathscr{S})$. Then, there exist $C_{\mathrm{Ent}}, C_{\mathrm{LSI}}$ satisfying*

$$\frac{1}{2\ln(1+\mathrm{e}^2)}\, C_{\mathrm{Ent}} \leq C_{\mathrm{LSI}} \leq 4\, C_{\mathrm{Ent}}$$

such that the following statements are equivalent:

(i) For all $A, B \subset \mathscr{S}$ disjoint with

$$\nu[A] \leq \tfrac{1}{2} \quad \textit{and} \quad \nu[B] \geq \tfrac{1}{2} \qquad \textit{holds} \qquad \nu[A] \ln\!\left(1 + \frac{\mathrm{e}^2}{\nu[A]}\right) \leq C_{\mathrm{Ent}} \operatorname{cap}(A, B).$$

(ii) The mixed logarithmic Sobolev inequality holds, that is

$$\mathrm{Ent}_\nu[f] \leq C_{\mathrm{LSI}}\, \mathscr{E}(f), \qquad \forall f \in \ell^2(\mu).$$

Proof. (i) $\Rightarrow$ (ii): We shift f according to its median m with respect to ν (cf. proof of Corollary 2.10) by applying (2.13) to $f - m$ and get

$$\mathrm{Ent}_\nu[f^2] \leq \big\|(f-m)^2\big\|_{\Phi,\nu,\mathrm{e}^2} \leq \big\|(f-m)_+^2\big\|_{\Phi,\nu,\mathrm{e}^2} + \big\|(f-m)_-^2\big\|_{\Phi,\nu,\mathrm{e}^2}.$$

The first implication follows by applying Proposition 2.5 and combining $\mathscr{E}((f-m)_+)$ and $\mathscr{E}((f-m)_-)$ as in the proof of Corollary 2.10.

(ii) $\Rightarrow$ (i): The converse statement follows exactly along the lines of the proof of Corollary 2.9 with the additional assumption that $\nu[B] \geq \frac{1}{2}$. □

3. Application to metastable Markov chains

In Section 3.1, we derive estimates and other technical tools based on the capacitary inequality as well the metastable assumption. The Sections 3.2 and 3.3 contain the main results on the asymptotic sharp estimates for the Poincaré and logarithmic Sobolev constants for metastable Markov chains, respectively.

Throughout this section we suppose that Assumption 1.3 holds.

3.1. A priori estimates.

In order to apply the definition of metastable sets, we first show that for any subset of the local valley $\mathscr{V}_i$ the hitting probability of the union of all metastable sets can be replaced by the hitting probability of any single set $M \in \mathscr{M}$.

Lemma 3.1. *For any $M_i \in \mathscr{M}$ and $A \subset \mathscr{V}_i \setminus M_i$,*

$$\mathbb{P}_{\mu_A}\big[\tau_{M_i} < \tau_A\big] \geq \frac{1}{|\mathscr{M}|} \mathbb{P}_{\mu_A}\big[\tau_{\bigcup_{j=1}^K M_j} < \tau_A\big]. \tag{3.1}$$

In particular,

$$\mathbb{P}_{\mu_A}\big[\tau_{M_i} < \tau_A\big] \geq \frac{1}{\varrho} \max_{M \in \mathscr{M}} \mathbb{P}_{\mu_M}\big[\tau_{\bigcup_{j=1}^K M_j \setminus M} < \tau_M\big]. \tag{3.2}$$

Proof. Since (3.2) is an immediate consequence of (3.1) and Definition 1.1, it suffices to prove (3.1). Since $\mathbb{P}_x[\tau_M < \tau_{\bigcup_{j=1}^K M_j \setminus M}] = \mathbb{P}_x[X(\tau_{\bigcup_{j=1}^K M_j}) = M]$ for any $M \in \mathscr{M}$ and $x \in \mathscr{S}$, we obtain

$$1 = \sum_{M \in \mathscr{M}} \mathbb{P}_x[\tau_M < \tau_{\bigcup_{j=1}^K M_j \setminus M}] \leq |\mathscr{M}|\, \mathbb{P}_x[\tau_{M_i} < \tau_{\bigcup_{j=1}^K M_j \setminus M_i}], \qquad \forall\, x \in \mathscr{V}_i.$$

Thus,

$$\mathbb{P}_{\nu_{A,B}}\big[\tau_{M_i} < \tau_{\bigcup_{j=1}^K M_j \setminus M_i}\big] \geq \frac{1}{|\mathscr{M}|}, \qquad \forall\, A \subset \mathscr{V}_i \setminus M_i, \tag{3.3}$$

where $\nu_{A,B}$ is the last-exit biased distribution as defined in (1.10) with $B = \bigcup_{j=1}^K M_j$. On the other hand, by using averaged renewal estimates that has been proven in [41, Lemma 1.24], we get that

$$\mathbb{P}_{\nu_{A,B}}\big[\tau_{M_i} < \tau_{\bigcup_{j=1}^K M_j \setminus M_i}\big] \leq \frac{\mathbb{P}_{\mu_A}\big[\tau_{M_i} < \tau_A\big]}{\mathbb{P}_{\mu_A}\big[\tau_{\bigcup_{j=1}^K M_j} < \tau_A\big]}. \tag{3.4}$$

By combining the estimates (3.3) and (3.4), (3.1) follows. □

The following lemma shows that the intersection of different local valleys has a negligible mass under the invariant distribution.

Lemma 3.2. *Suppose that $X := \mathscr{V}_k \cap \mathscr{V}_l \setminus (M_k \cup M_l)$ is non-empty. Then, it holds that*

$$\mu[X] \leq \varrho\, |\mathscr{M}| \min\big\{\mu[M_k], \mu[M_l]\big\}.$$

Proof. Without loss of generality assume that $\mu[M_k] \leq \mu[M_l]$. Notice that by (3.3), $h_{M_l,M_k}(x) \geq \mathbb{P}_x[\tau_{M_l} < \tau_{\bigcup_{j=1}^K M_j \setminus M_l}] \geq 1/|\mathscr{M}|$ for any $x \in X \subset \mathscr{V}_l \setminus M_l$. Therefore,

$$\operatorname{cap}(M_k, M_l) \;\geq\; \langle -L h_{M_l,M_k}, h_{X,M_k} \rangle_\mu \;=\; \langle h_{M_l,M_k}, -L h_{X,M_k} \rangle_\mu \;\geq\; \frac{1}{|\mathscr{M}|} \operatorname{cap}(X, M_k).$$

Thus,

$$\mu[X] \;\leq\; |\mathscr{M}| \, \frac{\operatorname{cap}(M_k, M_l)}{\mathbb{P}_{\mu_X}\bigl[\tau_{M_k} < \tau_X\bigr]} \;\overset{(3.2)}{\leq}\; \varrho |\mathscr{M}| \, \frac{\operatorname{cap}(M_k, M_l)}{\mathbb{P}_{\mu_{M_k}}\bigl[\tau_{\bigcup_{j=1}^K M_j \setminus M_k} < \tau_{M_k}\bigr]} \;\leq\; \varrho |\mathscr{M}| \, \mu[M_k],$$

which concludes the proof. □

The capacitary inequality combined with the definition of metastable sets yields that the harmonic functions, h_{M_i,M_j}, is almost constant on the valleys $\mathscr{S}_i$ and $\mathscr{S}_j$.

Lemma 3.3 (ℓ^p-norm estimate)**.** *For any $M_i \in \mathscr{M}$ and $f \in \ell^2(\mu)$ with $f(x) = 0$ for all $x \in M_i$,*

$$\mathrm{E}_{\mu_i}\bigl[f^2\bigr] \;\leq\; \frac{4\,\varrho}{\mu[\mathscr{S}_i]} \biggl(\max_{M \in \mathscr{M}} \mathbb{P}_{\mu_M}\bigl[\tau_{\bigcup_{j=1}^K M_j \setminus M} < \tau_M\bigr] \biggr)^{-1} \mathscr{E}(f). \tag{3.5}$$

In particular, for any $M_i, M_j \in \mathscr{M}$ with $i \neq j$,

$$\mathrm{E}_{\mu_i}\bigl[h_{M_j,M_i}^p\bigr] \;\leq\; \varrho \, \frac{p}{p-1} \, \min\biggl\{1, \frac{\mu[\mathscr{S}_j]}{\mu[\mathscr{S}_i]}\biggr\}, \qquad \forall\, p > 1, \tag{3.6}$$

and

$$\mathrm{E}_{\mu_i}\bigl[h_{M_j,M_i}\bigr] \;\leq\; \varepsilon \,+\, \varrho \, \ln(1/\varepsilon) \, \min\biggl\{1, \frac{\mu[\mathscr{S}_j]}{\mu[\mathscr{S}_i]}\biggr\}, \qquad \forall\, \varepsilon \in (0,1], \tag{3.7}$$

where h_{M_j,M_i} denotes the equilibrium potential of the pair (M_j, M_i).

Proof. First, notice that for any $A \subset \mathscr{V}_i \setminus M_i$

$$\mu_i[A] \;=\; \frac{\operatorname{cap}(A, M_i)}{\mu[\mathscr{S}_i] \, \mathbb{P}_{\mu_A}[\tau_{M_i} < \tau_A]} \;\overset{(3.2)}{\leq}\; \frac{\varrho}{\mu[\mathscr{S}_i]} \biggl(\max_{M \in \mathscr{M}} \mathbb{P}_{\mu_M}\bigl[\tau_{\bigcup_{j=1}^K M_j \setminus M} < \tau_M\bigr] \biggr)^{-1} \operatorname{cap}(A, M_i).$$

Thus, (3.5) follows from Proposition 2.5 by choosing $(\Phi, \Psi) = (\Phi_1, \Psi_1)$ as in Example 2.4 a). In the sequel, we aim to prove (3.6) and (3.7). For any $t \in [0,1]$ we write $A_t := \{x \in \mathscr{S} : h_{M_j,M_i} > t\}$ to denote the super level-sets of h_{M_j,M_i}, and set $h_t := h_{A_t,M_i}$. Then,

$$t \operatorname{cap}(A_t, M_i) \;\leq\; \langle -L h_t, h_{M_j,M_i} \rangle_\mu \;=\; \langle h_t, -L h_{M_j,M_i} \rangle_\mu \;=\; \operatorname{cap}(M_j, M_i). \tag{3.8}$$

Thus, for any $p > 1$, we obtain

$$\begin{aligned} \mathrm{E}_{\mu_i}\bigl[h_{M_j,M_i}^p\bigr] \;&=\; \int_0^1 p t^{p-1} \mu_i[A_t] \,\mathrm{d}t \\ &\leq\; \frac{\varrho}{\mu[\mathscr{S}_i]} \biggl(\max_{M \in \mathscr{M}} \mathbb{P}_{\mu_M}\bigl[\tau_{\bigcup_{j=1}^K M_j \setminus M} < \tau_M\bigr] \biggr)^{-1} \int_0^1 p t^{p-1} \operatorname{cap}(A_t, M_i) \,\mathrm{d}t. \end{aligned}$$

Since,

$$\max_{M\in\mathscr{M}} \mathbb{P}_{\mu_M}\big[\tau_{\bigcup_{j=1}^K M_j\setminus M} < \tau_M\big] \geq \max\big\{\mathbb{P}_{\mu_{M_i}}[\tau_{M_j} < \tau_{M_i}], \mathbb{P}_{\mu_{M_j}}[\tau_{M_i} < \tau_{M_j}]\big\}$$

we deduce that

$$\mathrm{E}_{\mu_i}\big[h^p_{M_j,M_i}\big] \overset{(3.8)}{\leq} \varrho \min\Big\{1, \frac{\mu[\mathscr{S}_j]}{\mu[\mathscr{S}_i]}\Big\} \int_0^1 p t^{p-2}\,\mathrm{d}t \;=\; \varrho \,\frac{p}{p-1} \min\Big\{1, \frac{\mu[\mathscr{S}_j]}{\mu[\mathscr{S}_i]}\Big\},$$

which concludes the proof of (3.6). Likewise, we obtain for any $\varepsilon \in (0,1]$ that

$$\mathrm{E}_{\mu_i}\big[h_{M_j,M_i}\big] \;=\; \varepsilon + \int_\varepsilon^1 \mu_i[A_t]\,\mathrm{d}t \overset{(3.8)}{\leq} \varepsilon + \varrho \min\Big\{1, \frac{\mu[\mathscr{S}_j]}{\mu[\mathscr{S}_i]}\Big\} \int_\varepsilon^1 t^{-1}\,\mathrm{d}t,$$

and (3.7) follows. □

The bound (3.7) of Lemma 3.3 provides the main ingredient for the proof of Proposition 1.12.

Proof of Proposition 1.12. Let $\mathscr{M}_i$ and $J \equiv J(i)$ be defined as in Proposition 1.12 and set $B = \bigcup_{j\in J} M_j$. By [11, Corollary 7.11] we have that

$$\mathbb{E}_{\nu_{M_i,B}}[\tau_B] \;=\; \frac{\mathrm{E}_\mu[h_{M_i,B}]}{\mathrm{cap}(M_i,B)} \;=\; \frac{\mu[\mathscr{S}_i]}{\mathrm{cap}(M_i,B)} \bigg(\mathrm{E}_{\mu_i}[h_{M_i,B}] + \sum_{j\neq i} \frac{\mu[\mathscr{S}_j]}{\mu[\mathscr{S}_i]}\,\mathrm{E}_{\mu_j}[h_{M_i,B}]\bigg).$$

In order to prove a lower bound, we neglect the last term in the bracket above. Since $\mathbb{P}_x[\tau_{\bigcup_{j\in J} M_j} < \tau_{M_i}] \leq \sum_{j\in J} \mathbb{P}_x[\tau_{M_j} < \tau_{M_i}]$, Lemma 3.3 implies with $\varepsilon = \varrho$

$$\mathrm{E}_{\mu_i}[h_{M_i,B}] \;=\; 1 - \mathrm{E}_{\mu_i}[h_{B,M_i}] \;\geq\; 1 - \sum_{j\in J} \mathrm{E}_{\mu_i}[h_{M_j,M_i}] \overset{(3.7)}{\geq} 1 - |\mathscr{M}|\,\varrho\big(1 + \ln 1/\varrho\big).$$

Hence, we conclude that

$$\mathbb{E}_{\nu_{M_i,B}}[\tau_B] \geq \frac{\mu[\mathscr{S}_i]}{\mathrm{cap}(M_i,B)} \big(1 - |\mathscr{M}|(\varrho + \varrho \ln 1/\varrho)\big).$$

Concerning the upper bound, recall that by assumption $\mu[\mathscr{S}_j]/\mu[\mathscr{S}_i] \leq \delta$ for all $j \notin J \cup \{i\}$. Thus, by Lemma 3.3 with $\varepsilon = \varrho/C_{\mathrm{ratio}}$, we get

$$\sum_{j\neq i} \frac{\mu[\mathscr{S}_j]}{\mu[\mathscr{S}_i]}\,\mathrm{E}_{\mu_j}[h_{M_i,B}] \;\leq\; |\mathscr{M}|\,\delta + \sum_{j\in J} \frac{\mu[\mathscr{S}_j]}{\mu[\mathscr{S}_i]}\,\mathrm{E}_{\mu_j}[h_{M_i,M_j}] \overset{(3.7)}{\leq} |\mathscr{M}|\Big(\delta + \varrho + \varrho \ln \frac{C_{\mathrm{ratio}}}{\varrho}\Big).$$

Since $\mathrm{E}_{\mu_i}[h_{M_i,B}] \leq 1$, the proof concludes with the estimate

$$\mathbb{E}_{\nu_{M_i,B}}[\tau_B] \;\leq\; \frac{\mu[\mathscr{S}_i]}{\mathrm{cap}(M_i,B)} \big(1 + |\mathscr{M}|\big(\delta + \varrho + \varrho \ln(C_{\mathrm{ratio}}/\varrho)\big)\big).$$

□

Let us define neighborhoods of the metastable sets in terms of level sets of harmonic functions. Therefore, we consider two non-empty, disjoint subset $\mathscr{A}, \mathscr{B} \subset \mathscr{M}$ of the set of metastable sets and let $I_\mathscr{A}, I_\mathscr{B} \subset \{1,\dots,K\}$ be such that $\mathscr{A} = \{M_i :$

$i \in I_{\mathscr{A}}\}$ and $\mathscr{B} = \{M_i : i \in I_{\mathscr{B}}\}$. Further, set $A = \bigcup_{M\in\mathscr{A}} M$ and $B = \bigcup_{M\in\mathscr{B}} M$. For $\delta \in (0,1)$ define the harmonic neighborhood of A relative to B by

$$\mathscr{U}_A(\delta,B) := \Big\{x \in \bigcup_{i\in I_{\mathscr{A}}} \mathscr{S}_i : h_{A,B}(x) \geq 1-\delta\Big\}. \tag{3.9}$$

The following lemma shows that the capacity of $(\mathscr{U}_A(\delta,B), \mathscr{U}_B(\delta,A))$ is comparable to the capacity of (A,B).

Lemma 3.4. *Let $M_i \in \mathscr{M}$ and $\mathscr{B} \subset \mathscr{M} \setminus \{M_i\}$. Then, for any $\delta \in (0,1/2)$,*

$$1-2\delta \;\leq\; \frac{\mathrm{cap}(M_i,B)}{\mathrm{cap}(\mathscr{U}_{M_i}(\delta,B), \mathscr{U}_B(\delta,M_i))} \;\leq\; 1. \tag{3.10}$$

Moreover, for $X := \mathscr{S}_i \setminus \mathscr{U}_{M_i}(\delta,B)$,

$$\mu[X] \;\leq\; \varrho\, \delta^{-1}\, \mu[M_i]. \tag{3.11}$$

Proof. Since $M_i \subset \mathscr{U}_{M_i}(\delta,B)$ and $B \subset \mathscr{U}_B(\delta,M_i)$ by definition, the upper bound in (3.10) follows from the monotonicity of the capacity, see (1.8). In order to prove the lower bound in (3.10), notice that

$$h_{M_i,B}(x) \;\geq\; 1-\delta, \quad \forall\, x \in \mathscr{U}_{M_i}(\delta,B) \qquad \text{and} \qquad h_{M_i,B}(x) \;\leq\; \delta, \quad \forall\, x \in \mathscr{U}_B(\delta,M_i).$$

Thus, by using the symmetry of $-L$ in $\ell^2(\mu)$, we obtain

$$\begin{aligned}\mathrm{cap}(M_i,B) \;&=\; \langle -Lh_{M_i,B}, h_{\mathscr{U}_{M_i}(\delta),\mathscr{U}_B(\delta)}\rangle_\mu \\ &=\; \langle h_{M_i,B}, -Lh_{\mathscr{U}_{M_i}(\delta),\mathscr{U}_B(\delta)}\rangle_\mu \;\geq\; \mathrm{cap}(\mathscr{U}_{M_i}(\delta), \mathscr{U}_B(\delta))\big(1-2\delta\big).\end{aligned}$$

The proof of (3.11) is similar to the one of Lemma 3.2. Since $h_{M_i,B}(x) \leq 1-\delta$ for any $x \in X = \mathscr{S}_i \setminus \mathscr{U}_{M_i}(\delta,B)$, we get

$$\begin{aligned}\mathrm{cap}(M_i,B) \;&\geq\; \langle -Lh_{B,M_i}, h_{X,M_i}\rangle_\mu \;=\; \langle h_{B,M_i}, -Lh_{X,M_i}\rangle_\mu \\ &\geq\; \delta\;\mathrm{cap}(X,M_i) \overset{(3.2)}{\geq} \frac{\delta\,\mu[X]}{\varrho}\; \mathbb{P}_{\mu_{M_i}}\big[\tau_B < \tau_{M_i}\big].\end{aligned}$$

Thus, the assertion follows from (1.7). □

3.2. **Poincaré inequality.** In this section we denote by c a numerical finite constant, which may change from line to line.

Theorem 3.5. *Suppose that Assumption 1.8 and 1.10 i) hold. Then, it holds that*

$$C_{\mathrm{PI}} \;\geq\; \max_{\substack{i,j\in\{1,\ldots,K\}\\ i\neq j}} \frac{\mu[\mathscr{S}_i]\,\mu[\mathscr{S}_j]}{\mathrm{cap}(M_i,M_j)}\big(1-c\sqrt{\varrho}\big), \tag{3.12}$$

$$C_{\mathrm{PI}} \;\leq\; \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{K} \frac{\mu[\mathscr{S}_i]\,\mu[\mathscr{S}_j]}{\mathrm{cap}(M_i,M_j)}\Big(1+c\sqrt{C_{\mathrm{PI},\mathscr{M}}(\varrho+\eta)}\Big). \tag{3.13}$$

Remark 3.6. It is possible to formulate a result with asymptotically matching upper and lower bounds for C_{PI} under suitable non-degeneracy assumption. These essentially demand that one of the term in the right hand side of (3.13) dominates the others.

Let $\mathscr{G} := \sigma(\mathscr{S}_i : i = 1, \dots, K)$ be the σ-algebra generated by the sets of the metastable partition. Obviously, $\mathscr{G} \subset \mathscr{F}$. We denote by $\mathrm{E}_\mu[f \mid \mathscr{G}]$ the conditional expectation given $\mathscr{G}$. In order to prove Theorem 3.5 we use again the projection property of the conditional expectation to further split the variance $\mathrm{Var}_\mu[\mathrm{E}_\mu[f \mid \mathscr{F}]]$ into the local variances and the mean difference

$$\begin{aligned}\mathrm{Var}_\mu&\big[\,\mathrm{E}_\mu[f \mid \mathscr{F}]\big]\\ &= \sum_{i=1}^K \mu[\mathscr{S}_i]\,\mathrm{Var}_{\mu_i}\big[\,\mathrm{E}[f \mid \mathscr{F}]\big] + \frac{1}{2}\sum_{i,j=1}^K \mu[\mathscr{S}_i]\mu[\mathscr{S}_j]\Big(\mathrm{E}_{\mu_i}[f] - \mathrm{E}_{\mu_j}[f]\Big)^2. \qquad (3.14)\end{aligned}$$

Therewith, the proof of Theorem 3.5 consists in bounding both the local variances and the mean difference in terms of the Dirichlet form. Bounding the local variances is established by local Poincaré inequalities, which are a consequence of Lemma 3.3.

Lemma 3.7 (Local Poincaré inequality). *Suppose that Assumption 1.10 i) is satisfied. Then, for any $f \in \ell^2(\mu)$ and $i \in \{1, \dots, K\}$,*

$$\begin{aligned}\mathrm{Var}_{\mu_i}\big[\mathrm{E}_\mu[f \mid \mathscr{F}]\big] &\le \mathrm{E}_{\mu_i}\Big[\big(\mathrm{E}_\mu[f \mid \mathscr{F}] - \mathrm{E}_{\mu_{M_i}}[f]\big)^2\Big]\\ &\le c\,\frac{C_{\mathrm{PI},\mathscr{M}}\,\varrho}{\mu[\mathscr{S}_i]}\,\Big(\max_{M \in \mathscr{M}} \mathbb{P}_{\mu_M}\big[\tau_{\bigcup_{j=1}^K M_j \setminus M} < \tau_M\big]\Big)^{-1} \mathscr{E}(f). \qquad (3.15)\end{aligned}$$

Proof. By noting that $\mathrm{Var}_{\mu_i}[\mathrm{E}_\mu[f \mid \mathscr{F}]] = \min_{a \in \mathbb{R}} \mathrm{E}_{\mu_i}[(\mathrm{E}_\mu[f \mid \mathscr{F}] - a)^2]$, the first estimate in (3.15) is immediate. Notice that the function $x \mapsto \mathrm{E}_\mu[f \mid \mathscr{F}](x) - \mathrm{E}_{\mu_{M_i}}[f]$ vanishes on M_i. Hence, by (3.5) we obtain

$$\mathrm{E}_{\mu_i}\Big[(\mathrm{E}_\mu[f \mid \mathscr{F}] - \mathrm{E}_{\mu_{M_i}}[f])^2\Big] \le \frac{4\varrho}{\mu[\mathscr{S}_i]}\Big(\max_{M \in \mathscr{M}} \mathbb{P}_{\mu_M}\big[\tau_{\bigcup_{j=1}^K M_j \setminus M} < \tau_M\big]\Big)^{-1} \mathscr{E}\big(\mathrm{E}_\mu[f \mid \mathscr{F}]\big).$$

Thus, we are left with bounding $\mathscr{E}\big(\mathrm{E}_\mu[f \mid \mathscr{F}]\big)$ from above by $\mathscr{E}(f)$. For any $\delta > 0$ by Young's inequality, that reads $|ab| \le \delta a^2/2 + b^2/(2\delta)$, we get for any $x, y \in \mathscr{S}$

$$\begin{aligned}\big(\mathrm{E}_\mu&[f \mid \mathscr{F}](x) - \mathrm{E}_\mu[f \mid \mathscr{F}](y)\big)^2\\ &\le \big(1 + 2\delta\big)\big(f(x) - f(y)\big)^2 + \Big(2 + \frac{1}{\delta}\Big) \sum_{z \in \{x,y\}} \big(f(z) - \mathrm{E}_\mu[f \mid \mathscr{F}](z)\big)^2.\end{aligned}$$

Recall that $\mathrm{E}_\mu[f \mid \mathscr{F}](x) = \mathrm{E}_{\mu_{M_i}}[f]$ for any $x \in M_i$. Since $f(x) - \mathrm{E}_\mu[f \mid \mathscr{F}](x) = 0$ for any $x \in \mathscr{S} \setminus \bigcup_{i=1}^K M_i$, we obtain

$$\mathscr{E}\big(\mathrm{E}_\mu[f \mid \mathscr{F}]\big) \le \big(1 + 2\delta\big)\,\mathscr{E}(f) + \Big(2 + \frac{1}{\delta}\Big)\sum_{i=1}^K \mu[M_i]\,\mathrm{Var}_{\mu_{M_i}}[f].$$

Since $\mathrm{Var}_{\mu_{M_i}}[f] \le C_{\mathrm{PI},i}\,\mathcal{E}(f)$ for any $i = 1,\ldots,K$, the assertion (3.15) follows by choosing $\delta = \sqrt{2C_{\mathrm{PI},\mathcal{M}}}$. $\square$

Lemma 3.8 (Mean difference estimate)**.** *Let Assumptions 1.8 and 1.10 i) be satisfied. Then, for any $f \in \ell^2(\mu)$ and $M_i, M_j \in \mathcal{M}$ with $i \neq j$ it holds that*

$$\Big(\mathrm{E}_{\mu_i}[f] - \mathrm{E}_{\mu_j}[f]\Big)^2 \le \frac{\mathcal{E}(f)}{\mathrm{cap}(M_i, M_j)}\Big(1 + c\sqrt{C_{\mathrm{PI},\mathcal{M}}(\varrho + \eta)}\Big).$$

Proof. For $M_i, M_j \in \mathcal{M}$ with $i \neq j$ let ν_{M_i,M_j} be the last-exit biased distribution as defined in (1.10) and denote by $g_{i,j} := \nu_{M_i,M_j}/\mu_{M_i}$ the relative density of ν_{M_i,M_j} with respect to μ_{M_i}. Then, it holds that

$$\begin{aligned}\mathrm{E}_{\mu_i}[f] &= \mathrm{E}_{\mu_i}\big[\mathrm{E}_{\mu}[f \mid \mathcal{F}]\big]\\ &= \mathrm{E}_{\nu_{M_i,M_j}}[f] + \mathrm{E}_{\mu_i}\Big[\mathrm{E}_{\mu}[f \mid \mathcal{F}] - \mathrm{E}_{\mu_{M_i}}[f]\Big] - \mathrm{E}_{\mu_{M_i}}\big[(g_{i,j} - 1)f\big].\end{aligned}$$

Thus, by applying Young's inequality, we obtain for any $\delta > 0$ and $f \in \ell^2(\mu)$,

$$\begin{aligned}\Big(\mathrm{E}_{\mu_i}[f] - \mathrm{E}_{\mu_j}[f]\Big)^2 &\le (1+\delta)\Big(\mathrm{E}_{\nu_{M_i,M_j}}[f] - \mathrm{E}_{\nu_{M_j,M_i}}[f]\Big)^2\\ &\quad + 2\Big(1 + \frac{1}{\delta}\Big)\sum_{k \in \{i,j\}} \mathrm{E}_{\mu_k}\Big[\mathrm{E}_{\mu}[f \mid \mathcal{F}] - \mathrm{E}_{\mu_{M_k}}[f]\Big]^2\\ &\quad + 2\Big(1 + \frac{1}{\delta}\Big)\Big(\mathrm{E}_{\mu_{M_i}}\big[(g_{i,j} - 1)f\big]^2 + \mathrm{E}_{\mu_{M_j}}\big[(g_{j,i} - 1)f\big]^2\Big).\end{aligned}$$

Let h_{M_i,M_j} be the equilibrium potential of the pair (M_i, M_j). Observe that a summation by parts together with an application of the Cauchy-Schwarz inequality yields

$$\Big(\mathrm{E}_{\nu_{M_i,M_j}}[f] - \mathrm{E}_{\nu_{M_i,M_j}}[f]\Big)^2 \le \frac{\mathcal{E}(f)}{\mathrm{cap}(M_i, M_j)}.$$

Recall that the function $x \mapsto \mathrm{E}_{\mu}[f \mid \mathcal{F}](x) - \mathrm{E}_{\mu_{M_i}}[f]$ vanishes on M_i. Thus, (3.15) implies that

$$\mathrm{E}_{\mu_i}\Big[\big(\mathrm{E}_{\mu}[f \mid \mathcal{F}] - \mathrm{E}_{\mu_{M_i}}[f]\big)^2\Big] \le c\, C_{\mathrm{PI},\mathcal{M}}\, \varrho\, \frac{\mathcal{E}(f)}{\mathrm{cap}(M_i, M_j)},$$

where we used that $\max_{M \in \mathcal{M}} \mathbb{P}_{\mu_M}\big[\tau_{\bigcup_{j=1}^K M_j \setminus M} < \tau_M\big] \ge \mathrm{cap}(M_i, M_j)/\mu[M_i]$. Further, the covariance between $g_{i,j}$ and f, thanks to Assumptions 1.8 and 1.10 i), is bounded from above by

$$\mathrm{E}_{\mu_{M_i}}\big[(g_{i,j} - 1)f\big]^2 \le \mathrm{Var}_{\mu_{M_i}}[g_{i,j}]\,\mathrm{Var}_{\mu_{M_i}}[f] \le \eta\, \mu[M_i]\, C_{\mathrm{PI},i}\, \frac{\mathcal{E}(f)}{\mathrm{cap}(M_i, M_j)}.$$

By combining the estimates above and choosing $\delta = \sqrt{C_{\mathrm{PI},\mathcal{M}}(\varrho + \eta)}$, we obtain the assertion. $\square$

A combination of the splitting (3.14) with the Lemmas 3.7 and 3.8 gives the upper bound (3.13) of Theorem 3.5. The proof is complemented by a suitable test function yielding the lower bound (3.12).

Proof of Theorem 3.5. The lower bound of C_{PI} is an immediate consequence of the variational definition of C_{PI}, cf. (1.11). Indeed, by choosing the equilibrium potential h_{M_i,M_j} for any $M_i, M_j \in \mathscr{M}$ with $i \neq j$ as a test function, we deduce from (1.13) and (3.14) that

$$\begin{aligned} \mathrm{Var}_\mu\big[h_{M_i,M_j}\big] &\geq \mu[\mathscr{S}_i]\,\mu[\mathscr{S}_j]\,\big(\mathrm{E}_{\mu_i}[h_{M_i,M_j}] - \mathrm{E}_{\mu_j}[h_{M_i,M_j}]\big)^2 \\ &= \mu[\mathscr{S}_i]\,\mu[\mathscr{S}_j]\,\big(1 - \mathrm{E}_{\mu_i}[h_{M_j,M_i}] - \mathrm{E}_{\mu_j}[h_{M_i,M_j}]\big)^2. \end{aligned}$$

Thus, in view of (3.7), we obtain that $\mathrm{Var}_\mu[f] \geq \mu[\mathscr{S}_i]\mu[\mathscr{S}_j]\big(1-8\varrho\big)^2$. Since $\mathscr{E}(h_{M_i,M_j}) = \mathrm{cap}(M_i, M_j)$, (3.12) follows by optimizing over all $M_i \neq M_j \in \mathscr{M}$.

For the upper bound, observe that by using (1.7) and (1.8), it holds that

$$\begin{aligned} &\Big(\max_{M\in\mathscr{M}} \mathbb{P}_{\mu_M}\big[\tau_{\bigcup_{j=1}^K M_j\setminus M} < \tau_M\big]\Big)^{-1} \\ &\quad\leq \frac{1}{|\mathscr{M}|-1} \sum_{\substack{i,j=1\\ i\neq j}}^K \frac{\mu[\mathscr{S}_i]}{\mathbb{P}_{\mu_{M_j}}\big[\tau_{M_i} < \tau_{M_j}\big]} \leq \frac{1}{|\mathscr{M}|-1} \sum_{\substack{i,j=1\\ i\neq j}}^K \frac{\mu[\mathscr{S}_i]\,\mu[\mathscr{S}_j]}{\mathrm{cap}(M_i,M_j)}. \end{aligned} \tag{3.16}$$

Hence, by an application of Lemma 3.7, it follows that

$$\sum_{i=1}^K \mu[\mathscr{S}_i]\,\mathrm{Var}_{\mu_i}\big[\mathrm{E}_\mu[f \mid \mathscr{F}]\big] \overset{(3.15)}{\leq} c\,C_{\mathrm{PI},\mathscr{M}}\,\varrho\,\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^K \frac{\mu[\mathscr{S}_i]\,\mu[\mathscr{S}_j]}{\mathrm{cap}(M_i,M_j)}\,\mathscr{E}(f).$$

Thus, a combination of (1.13) and (3.14) together with Lemma 3.8 yields (3.13) up to an additive factor $C_{\mathrm{PI},\mathscr{M}}$. To bound this additive error term, notice that

$$C_{\mathrm{PI},\mathscr{M}} \leq C_{\mathrm{PI},\mathscr{M}}\,\varrho\,\Big(\max_{M\in\mathscr{M}} \mathbb{P}_{\mu_M}\big[\tau_{\bigcup_{j=1}^K M_j\setminus M} < \tau_M\big]\Big)^{-1} \overset{(3.16)}{\leq} \frac{C_{\mathrm{PI},\mathscr{M}}\,\varrho}{|\mathscr{M}|-1} \sum_{\substack{i,j=1\\ i\neq j}}^K \frac{\mu[\mathscr{S}_i]\,\mu[\mathscr{S}_j]}{\mathrm{cap}(M_i,M_j)},$$

which shows that $C_{\mathrm{PI},\mathscr{M}}$ can be absorbed into the right hand side of (3.13). □

3.3. **Logarithmic Sobolev inequality.** In this subsection we focus on sharp estimates of the logarithmic Sobolev constant in the context of metastable Markov chains. Again, we denote by c a numerical finite constant, which may change from line to line.

Theorem 3.9. *Suppose that the Assumptions 1.8, 1.10 and (1.18) hold. Then, it holds that*

$$C_{\mathrm{LSI}} \geq \max_{\substack{i,j\in\{1,\ldots,K\}\\ i\neq j}} \frac{\mu[\mathscr{S}_i]\,\mu[\mathscr{S}_j]}{\Lambda(\mu[\mathscr{S}_i],\mu[\mathscr{S}_j])}\,\frac{1}{\mathrm{cap}(M_i,M_j)}\big(1-c\sqrt{\varrho}\big)^2 \tag{3.17}$$

$$C_{\mathrm{LSI}} \leq \sum_{\substack{i,j=1\\ i\neq j}}^K \frac{\mu[\mathscr{S}_i]\,\mu[\mathscr{S}_j]}{\Lambda(\mu[\mathscr{S}_i],\mu[\mathscr{S}_j])}\,\frac{1}{\mathrm{cap}(M_i,M_j)}\Big(1+c\sqrt{C_{\mathrm{mass}}\,C_{\mathrm{LSI},\mathscr{M}}(\varrho+\eta)}\Big). \tag{3.18}$$

In order to proof Theorem 3.9, we decompose the entropy $\operatorname{Ent}_{\mu}\big[\,\mathrm{E}_{\mu}[f^2\,|\,\mathscr{F}]\big]$ in (1.14) into the local entropies within the sets $\mathscr{S}_1,\dots,\mathscr{S}_K$ and the macroscopic entropy

$$\operatorname{Ent}_{\mu}\big[\mathrm{E}_{\mu}[f^2\,|\,\mathscr{F}]\big] \;=\; \sum_{i=1}^{K} \mu[\mathscr{S}_i]\,\operatorname{Ent}_{\mu_i}\big[\mathrm{E}_{\mu}[f^2\,|\,\mathscr{F}]\big] + \operatorname{Ent}_{\mu}\big[\mathrm{E}_{\mu}[f^2\,|\,\mathscr{G}]\big].$$

In the next lemma we derive an upper bound on the local entropies.

Lemma 3.10 (Local logarithmic Sobolev inequality)**.** *Let Assumption 1.10 i) be satisfied, and assume that $C_{\mathrm{mass}} < \infty$. Then, for any $f \in \ell^2(\mu)$ and $i \in \{1,\dots,K\}$,*

$$\operatorname{Ent}_{\mu_i}\big[\mathrm{E}_{\mu}[f^2\,|\,\mathscr{F}]\big] \;\le\; c\,\frac{C_{\mathrm{mass}}\,C_{\mathrm{PI},\mathscr{M}}\,\varrho}{\mu[\mathscr{S}_i]}\,\Big(\max_{M\in\mathscr{M}} \mathbb{P}_{\mu_M}\big[\tau_{\bigcup_{j=1}^K M_j\setminus M} < \tau_M\big]\Big)^{-1} \mathscr{E}(f). \tag{3.19}$$

Proof. First, notice that for any $A \subset \mathscr{S}\setminus M_i$

$$\begin{aligned}\mu_i[A]\,\ln\Big(1+\frac{\mathrm{e}^2}{\mu_i[A]}\Big) \;&\le\; \frac{\max_{x\in\mathscr{S}_i}\ln(1+\mathrm{e}^2/\mu_i(x))}{\mu[\mathscr{S}_i]\,\operatorname{cap}(A\cap\mathscr{S}_i, M_i)}\,\operatorname{cap}(A, M_i)\\ &\overset{(3.2)}{\le}\; C_{\mathrm{mass}}\,\frac{\varrho}{\mu[\mathscr{S}_i]}\,\Big(\max_{M\in\mathscr{M}} \mathbb{P}_{\mu_M}\big[\tau_{\bigcup_{j=1}^K M_j\setminus M} < \tau_M\big]\Big)^{-1}\operatorname{cap}(A, M_i).\end{aligned}$$

Since the function $x \mapsto \mathrm{E}_{\mu}[f^2\,|\,\mathscr{F}](x)$ is constant on M_i, Corollary 2.9 implies that

$$\operatorname{Ent}_{\mu_i}\big[\mathrm{E}_{\mu}[f^2\,|\,\mathscr{F}]\big] \;\le\; C_{\mathrm{mass}}\,\frac{4\varrho}{\mu[\mathscr{S}_i]}\,\Big(\max_{M\in\mathscr{M}} \mathbb{P}_{\mu_M}\big[\tau_{\bigcup_{j=1}^K M_j\setminus M} < \tau_M\big]\Big)^{-1}\mathscr{E}\Big(\sqrt{\mathrm{E}_{\mu}[f^2\,|\,\mathscr{F}]}\Big).$$

Thus, we are left with bounding $\mathscr{E}\big(\sqrt{\mathrm{E}_{\mu}[f^2\,|\,\mathscr{F}]}\big)$ from above with $\mathscr{E}(f)$. Applying Young's inequality, we get, for any $\delta > 0$ and $x, y \in \mathscr{S}$,

$$\begin{aligned}&\Big(\sqrt{\mathrm{E}_{\mu}[f^2\,|\,\mathscr{F}](x)} - \sqrt{\mathrm{E}_{\mu}[f^2\,|\,\mathscr{F}](y)}\Big)^2\\ &\quad\le\; \big(1+2\delta\big)\big(|f(x)|-|f(y)|\big)^2 + \Big(2+\frac{1}{\delta}\Big)\sum_{z\in\{x,y\}}\Big(|f(z)|-\sqrt{\mathrm{E}_{\mu}[f^2\,|\,\mathscr{F}](z)}\Big)^2.\end{aligned}$$

Since $|f(z)|-\sqrt{\mathrm{E}_{\mu}[f^2\,|\,\mathscr{F}](z)} = 0$ for any $z \in \mathscr{S}\setminus\bigcup_{i=1}^K M_i$ and $\mathrm{E}_{\mu}[f\,|\,\mathscr{F}](x) = \mathrm{E}_{\mu_{M_i}}[f]$ for any $x \in M_i$, we obtain

$$\mathscr{E}\Big(\sqrt{\mathrm{E}_{\mu}[f^2\,|\,\mathscr{F}]}\Big) \;\le\; \big(1+2\delta\big)\,\mathscr{E}(|f|) + 2\Big(2+\frac{1}{\delta}\Big)\sum_{i=1}^{K}\mu[M_i]\,\operatorname{Var}_{\mu_{M_i}}[f],$$

where we additionally exploited the fact that, by Jensens' inequality,

$$\mathrm{E}_{\mu_{M_i}}\Big[\Big(|f|-\sqrt{\mathrm{E}_{\mu_{M_i}}[f^2]}\Big)^2\Big] \;\le\; 2\,\operatorname{Var}_{\mu_{M_i}}[f].$$

Since $\operatorname{Var}_{\mu_{M_i}}[f] \le C_{\mathrm{PI},i}\,\mathscr{E}(f)$ for any $i = 1,\dots,K$ and $\mathscr{E}(|f|) \le \mathscr{E}(f)$, (3.19) follows by choosing $\delta = \sqrt{2C_{\mathrm{PI},\mathscr{M}}}$. □

Proof of Theorem 3.9. In view of the variational definition of C_{LSI}, cf. (1.12), (3.17) we follow from the construction of a suitable test function. For any $M_i, M_j \in \mathscr{M}$ with $i \neq j$, $\delta \in [0, 1/2)$ and $g : \{i, j\} \to \mathbb{R}$ set

$$f(x) := g(i)\, h_{\mathscr{U}_{M_i}(\delta), \mathscr{U}_{M_j}(\delta)}(x) + g(j)\, h_{\mathscr{U}_{M_j}(\delta), \mathscr{U}_{M_i}(\delta)}(x),$$

$\mathscr{U}_{M_i}(\delta) \equiv \mathscr{U}_{M_i}(\delta, M_j)$ and $\mathscr{U}_{M_j}(\delta) \equiv \mathscr{U}_{M_j}(\delta, M_i)$ are the δ-neighborhoods of M_i and M_j as defined in (3.9). Then, by Lemma 3.4,

$$\mathscr{E}(f) = \big(g(i) - g(j)\big)^2 \operatorname{cap}(\mathscr{U}_{M_i}(\delta), \mathscr{U}_{M_j}(\delta)) \overset{(3.10)}{\leq} \big(g(i) - g(j)\big)^2 \frac{\operatorname{cap}(M_i, M_j)}{1 - 2\delta}.$$

Further, notice that $a \ln a - a \ln b - a + b \geq 0$ for all $a, b > 0$. Thus,

$$\begin{aligned}
\operatorname{Ent}_\mu[f^2] &= \min_{c>0} \mathrm{E}_\mu\big[f^2 \ln f^2 - f^2 \ln c - f^2 + c\big] \\
&\geq \min_{c>0} \mathrm{E}_\mu\big[f^2 \ln f^2 - f^2 \ln c - f^2 + c)\, \mathbb{1}_{\mathscr{U}_{M_i}(\delta) \cup \mathscr{U}_{M_j}(\delta)}\big] \\
&\overset{(3.11)}{\geq} \big(\mu[\mathscr{S}_i] + \mu[\mathscr{S}_j]\big)\big(1 - \varrho\, \delta^{-1}\big) \operatorname{Ent}_{\mathrm{Ber}(p)}[g^2],
\end{aligned}$$

where $\mathrm{Ber}(p) \in \mathscr{P}(\{i, j\})$ denotes the Bernoulli measure on the two-point space $\{i, j\}$ with success probability $p = 1 - q = \mu[\mathscr{S}_i] / (\mu[\mathscr{S}_i] + \mu[\mathscr{S}_j])$. This yields

$$C_{\mathrm{LSI}} \geq \frac{\operatorname{Ent}_\mu[f^2]}{\mathscr{E}(f)} = \frac{\mu[\mathscr{S}_{M_i}] + \mu[\mathscr{S}_{M_j}]}{\operatorname{cap}(M_i, M_j)} (1 - 2\delta)\big(1 - \varrho\, \delta^{-1}\big) \frac{\operatorname{Ent}_{\mathrm{Ber}(p)}[g^2]}{(g(i) - g(j))^2},$$

for any $g : \{i, j\} \to \mathbb{R}$ with $g(i) \neq g(j)$. Recall that the logarithmic Sobolev constant for Bernoulli measures is explicitly known and given by

$$\sup\left\{\frac{\operatorname{Ent}_{\mathrm{Ber}(p)}[g^2]}{(g(i) - g(j))^2} : g(i) \neq g(j)\right\} = \frac{pq}{\Lambda(p, q)} = \frac{\mu[\mathscr{S}_i]\, \mu[\mathscr{S}_j]}{\Lambda(\mu[\mathscr{S}_i], \mu[\mathscr{S}_j])} \big(\mu[\mathscr{S}_i] + \mu[\mathscr{S}_j]\big).$$

This was found in [25] and independently in [21]. Thus, by choosing $\delta = \sqrt{\varrho}$, (3.17) follows.

Let us now address the upper bound. First, since $\Lambda(\mu[\mathscr{S}_i], \mu[\mathscr{S}_j]) \leq 1$ we deduce from Lemma 3.10 by following similar arguments as in (3.16) in the proof of Theorem 3.5 that

$$\sum_{i=1}^{K} \mu[\mathscr{S}_i] \operatorname{Ent}_{\mu_i}\big[\mathrm{E}[f^2 \mid \mathscr{F}]\big] \overset{(3.19)}{\leq} c\, C_{\mathrm{mass}}\, C_{\mathrm{PI}, \mathscr{M}}\, \varrho \sum_{\substack{i,j=1 \\ i \neq j}}^{K} \frac{\mu[\mathscr{S}_i]\, \mu[\mathscr{S}_j]}{\Lambda(\mu[\mathscr{S}_i], \mu[\mathscr{S}_j])} \frac{\mathscr{E}(f)}{\operatorname{cap}(M_i, M_j)}.$$

On the other hand, by [35, Corollary 2.8], we have that

$$\operatorname{Ent}_\mu\big[\mathrm{E}_\mu[f^2 \mid \mathscr{G}]\big] \leq \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{K} \frac{\mu[\mathscr{S}_i]\, \mu[\mathscr{S}_j]}{\Lambda(\mu[\mathscr{S}_i], \mu[\mathscr{S}_j])} \bigg(\sum_{k \in \{i,j\}} \operatorname{Var}_{\mu_k}[f] + \big(\mathrm{E}_{\mu_i}[f] - \mathrm{E}_{\mu_j}[f]\big)^2\bigg).$$

In view of the projection property of the conditional expectation together with (1.15) and (3.15)

$$\sum_{k\in\{i,j\}} \mathrm{Var}_{\mu_k}[f] = \sum_{k\in\{i,j\}} \Big(\mu_k[M_k]\,\mathrm{Var}_{\mu_{M_k}}[f] + \mathrm{Var}_{\mu_k}\big[\mathrm{E}_\mu[f\,|\,\mathscr{F}]\big]\Big) \le c C_{\mathrm{PI},\mathscr{M}} \frac{\varrho\,\mathscr{E}(f)}{\mathrm{cap}(M_i,M_j)}.$$

Thus, (3.18) follows up to the additive constant $C_{\mathrm{LSI},\mathscr{M}}$ by combining the estimates above and using Lemma 3.8. To bound the additive error term $C_{\mathrm{LSI},\mathscr{M}}$, notice

$$C_{\mathrm{LSI},\mathscr{M}} \overset{(3.16)}{\le} C_{\mathrm{LSI},\mathscr{M}}\,\varrho \sum_{\substack{i,j=1\\ i\neq j}}^{K} \frac{\mu[\mathscr{S}_i]\,\mu[\mathscr{S}_j]}{\Lambda(\mu[\mathscr{S}_i],\mu[cS_j])}\frac{1}{\mathrm{cap}(M_i,M_j)},$$

where we used that $\Lambda(\mu[\mathscr{S}_i],\mu[\mathscr{S}_j])\le 1$. This allows to absorb the additive constant $C_{\mathrm{LSI},\mathscr{M}}$ into the right hand side of (3.18). □

Proof of Theorem 1.13. For $K = 2$ (1.17) and (1.19) follows directly from Theorem 3.5 and Theorem 3.9. □

4. Random field Curie-Weiss model

The proof of Theorem 1.21 follows from the Theorems 3.5 and 3.9 after having established Propositions 1.20, 4.8, and 4.2 in each of the three following sections.

4.1. **Verification of ϱ-metastability.** In view of (1.5), estimates of hitting probabilities can be deduced from upper and lower bounds of the corresponding capacities. Based on the Dirichlet principle and a comparison argument for Dirichlet forms, our strategy is to compare the microscopic with suitable mesoscopic capacities via a coarse-graining. One direction of the comparison follows immediately from the Dirichlet principle.

For $\boldsymbol{A},\boldsymbol{B}\subset\Gamma^n$ disjoint set $A=\rho^{-1}(\boldsymbol{A})$ and $B=\rho^{-1}(\boldsymbol{B})$. Then, we can bound the *microscopic capacity* $\mathrm{cap}(A,B)$ from above by the *mesoscopic capacity* $\mathbf{cap}(\boldsymbol{A},\boldsymbol{B})$

$$\begin{aligned} \mathrm{cap}(A,B) &\overset{(1.9)}{\le} \inf_{g\in\mathscr{H}_{A,B}} \mathscr{E}(g\circ\rho) \\ &= \inf_{g\in\mathscr{H}_{A,B}} \frac{1}{2}\sum_{\boldsymbol{x},\boldsymbol{y}\in\Gamma^n} \boldsymbol{\mu}(\boldsymbol{x})\,\boldsymbol{r}(\boldsymbol{x},\boldsymbol{y})\big(g(\boldsymbol{x})-g(\boldsymbol{y})\big)^2 \\ &=: \mathbf{cap}(\boldsymbol{A},\boldsymbol{B}), \end{aligned} \tag{4.1}$$

where

$$\boldsymbol{r}(\boldsymbol{x},\boldsymbol{y}) := \frac{1}{\boldsymbol{\mu}(\boldsymbol{x})}\sum_{\sigma\in\rho^{-1}(\boldsymbol{x})}\mu(\sigma)\sum_{\sigma'\in\rho^{-1}(\boldsymbol{y})} p(\sigma,\sigma') \tag{4.2}$$

and $\mathscr{H}_{A,B} := \{g:\Gamma^n\to[0,1] : g|_{\boldsymbol{A}}=1, g|_{\boldsymbol{B}}=0\}$. Notice that the *mesoscopic transition probabilities* $(\boldsymbol{r}(\boldsymbol{x},\boldsymbol{y}) : \boldsymbol{x},\boldsymbol{y}\in\Gamma^n)$ are reversible with respect to $\boldsymbol{\mu}$. Recall that the metastable sets M_1,M_2 are defined as preimages under ρ of particular minima $\boldsymbol{m}_1,\boldsymbol{m}_2$ of F. Hence, an upper bound on the numerator in (1.3) follows from an upper bound on $\mathbf{cap}(\boldsymbol{m}_1,\boldsymbol{m}_2)$.

In the following lemma we show that the denominator in (1.3) can be as well expressed in terms of mesoscopic capacities.

Lemma 4.1. *For $n \geq 1$ let $\boldsymbol{B} \subset \Gamma^n$ be non-empty and set $B = \rho^{-1}(\boldsymbol{B})$. Further, define $\varepsilon(n) := 2h_\infty/n$. Then, for any $A \subset \mathscr{S} \setminus B$ and $N \geq n$,*

$$\mathbb{P}_{\mu_A}[\tau_B < \tau_A] \;\geq\; |\Gamma^n|^{-1}\, \mathrm{e}^{-4\beta\varepsilon(n)(2N+1)} \min_{\boldsymbol{x} \in \Gamma^n \setminus \boldsymbol{B}} \frac{\mathbf{cap}(\boldsymbol{x},\boldsymbol{B})}{\boldsymbol{\mu}(\boldsymbol{x})}. \tag{4.3}$$

Proof. Notice that the image process $(\rho(\sigma(t)) : t \geq 0)$ on Γ^n is in general not Markovian. For that reason, we introduce an additional Markov chain on $\mathscr{S}$ with the property that its image under ρ is Markov and the corresponding Dirichlet form is comparable to the original one with an error that can be controlled provided n is chosen large enough.

For fixed $n \geq 1$ let $(\bar{\sigma}(t) : t \geq 0)$ be a Markov chain in discrete-time on $\mathscr{S}$ with transition probabilities

$$\bar{p}(\sigma,\sigma') \;:=\; \frac{1}{N} \exp\bigl(-\beta N [E(\rho(\sigma')) - E(\rho(\sigma))]_+\bigr)\, \mathbb{1}_{|\sigma-\sigma'|_1 = 2}$$

and $\bar{p}(\sigma,\sigma) = 1 - \sum_{\sigma' \in \mathscr{S}} \bar{p}(\sigma,\sigma')$, which is reversible with respect to the random Gibbs measure

$$\bar{\mu}(\sigma) \;:=\; \bar{Z}^{-1} \exp\bigl(-\beta N E\bigl(\rho(\sigma)\bigr)\bigr)\, 2^{-N}, \qquad \sigma \in \mathscr{S}.$$

Let us denote the law of this process by $\bar{\mathbb{P}}$, and we write $\overline{\mathrm{cap}}(A,B)$ for the corresponding capacities. Likewise, let $\bar{\boldsymbol{\mu}} := \bar{\mu} \circ \rho^{-1}$ and $\bar{\boldsymbol{r}}$ analog to (4.2). Notice that

$$\mathrm{e}^{-2\beta\varepsilon(n)N} \;\leq\; \frac{\bar{\mu}(\sigma)}{\mu(\sigma)} \;\leq\; \mathrm{e}^{2\beta\varepsilon(n)N} \qquad \text{and} \qquad \mathrm{e}^{-2\beta\varepsilon(n)} \;\leq\; \frac{\bar{p}(\sigma,\sigma')}{p(\sigma,\sigma')} \;\leq\; \mathrm{e}^{2\beta\varepsilon(n)} \tag{4.4}$$

for any $\sigma, \sigma' \in \mathscr{S}$. On the other hand, for any $\boldsymbol{x}, \boldsymbol{y} \in \Gamma^n$ it holds that $\bar{p}(\sigma, \rho^{-1}(\boldsymbol{y})) = \bar{p}(\sigma', \rho^{-1}(\boldsymbol{y}))$ for every $\sigma, \sigma' \in \rho^{-1}(\boldsymbol{x})$. This ensures (see e.g. [17]) that the Markov chain $(\bar{\sigma}(t) : t \geq 0)$ is exactly lumpable, i.e. $(\rho(\bar{\sigma}(t)) : t \geq 0)$ is a Markov process on Γ^n with transition probabilities $\bar{\boldsymbol{r}}$ and reversible measure $\bar{\boldsymbol{\mu}}$. As a corollary of [11, Theorem 9.7] we obtain that, for $A = \rho^{-1}(\boldsymbol{a})$ and $B = \rho^{-1}(\boldsymbol{B})$ with $\{\boldsymbol{a}\}, \boldsymbol{B} \subset \Gamma^n$ disjoint,

$$\bar{\mathbb{P}}_\sigma[\tau_B < \tau_A] \;=\; \bar{\mathbb{P}}_{\sigma'}[\tau_B < \tau_A] \qquad \forall\, \sigma, \sigma' \in A. \tag{4.5}$$

In particular, $\overline{\mathrm{cap}}(A,B) = \overline{\mathbf{cap}}(\boldsymbol{a},\boldsymbol{B})$. By using a comparison of Dirichlet form, we deduce from (4.4) that for any $A, B \subset \mathscr{S}$

$$\mathrm{e}^{-2\beta\varepsilon(n)(N+1)} \;\leq\; \frac{\mathrm{cap}(A,B)}{\overline{\mathrm{cap}}(A,B)} \qquad \text{and} \qquad \mathrm{e}^{-2\beta\varepsilon(n)(N+1)} \;\leq\; \frac{\overline{\mathbf{cap}}(\boldsymbol{a},\boldsymbol{B})}{\mathbf{cap}(\boldsymbol{a},\boldsymbol{B})}. \tag{4.6}$$

Let us now address the proof of (4.3). For a given $\emptyset \neq \boldsymbol{B} \subset \Gamma^n$ set $B = \rho^{-1}(\boldsymbol{B})$ and let $A \subset \mathscr{S} \setminus B$ be arbitrary. Then, we can find $\{\boldsymbol{x}_k : k = 1, \ldots, L\} \subset \Gamma^n$ such that

$$A \cap \rho^{-1}(\boldsymbol{x}_k) \;\neq\; \emptyset \qquad \text{and} \qquad A \;\subset\; \bigcup_{k=1}^{L} \rho^{-1}(\boldsymbol{x}_k).$$

We set $X_k := \rho^{-1}(\boldsymbol{x}_k)$ and $A_k := A \cap X_k$ for $k \in \{1, \dots, L\}$ to lighten notation. Since

$$\overline{\mathrm{cap}}(A_k, B) \geq \sum_{\sigma \in A_k} \bar{\mu}(\sigma)\, \bar{\mathbb{P}}_\sigma\big[\tau_B < \tau_{X_k}\big] \overset{(4.5)}{=} \bar{\mu}[A_k]\, \frac{\overline{\mathrm{cap}}(X_k, B)}{\bar{\mu}[X_k]} = \bar{\mu}[A_k]\, \frac{\overline{\mathbf{cap}}(\boldsymbol{x}_k, \boldsymbol{B})}{\bar{\boldsymbol{\mu}}(\boldsymbol{x}_k)}$$

an application of (4.6) and (4.4) yields

$$\mathrm{cap}(A_k, B) \geq \mathrm{e}^{-2\beta\varepsilon(n)(N+1)}\, \bar{\mu}[A_k]\, \frac{\overline{\mathbf{cap}}(\boldsymbol{x}_k, \boldsymbol{B})}{\bar{\boldsymbol{\mu}}(\boldsymbol{x}_k)} \geq \mathrm{e}^{-4\beta\varepsilon(n)(2N+1)}\, \mu[A_k] \min_{\boldsymbol{x} \in \Gamma^n \setminus \boldsymbol{B}} \frac{\mathbf{cap}(\boldsymbol{x}, \boldsymbol{B})}{\boldsymbol{\mu}(\boldsymbol{x})}. \tag{4.7}$$

Thus,

$$\mathrm{cap}(A, B) \overset{(1.8)}{\geq} \frac{1}{L} \sum_{k=1}^{L} \mathrm{cap}(A_k, B) \overset{(4.7)}{\geq} \frac{1}{L}\, \mathrm{e}^{-4\beta\varepsilon(n)(2N+1)}\, \mu[A] \min_{\boldsymbol{x} \in \Gamma^n \setminus \boldsymbol{B}} \frac{\mathbf{cap}(\boldsymbol{x}, \boldsymbol{B})}{\boldsymbol{\mu}(\boldsymbol{x})}.$$

Since $L \leq |\Gamma^n|$, the assertion (4.3) follows. □

Proof of Proposition 1.20. For $n \geq 1$ let $M_1 = \rho^{-1}(\boldsymbol{m}_1)$ and $M_2 = \rho^{-1}(\boldsymbol{m}_2)$, where $\boldsymbol{m}_1, \boldsymbol{m}_2 \in \Gamma^n$ are local minima of F as in the assumptions of the Proposition. Then, by [5, Proposition 4.5, Corollary 4.6 and Proposition 3.1] there exists $C < \infty$ such that for any $N \geq N_0(h) \vee N_1(h)$

$$\mathbb{P}_{\mu_{M_2}}\big[\tau_{M_1} < \tau_{M_2}\big] \overset{(4.1)}{\leq} \frac{\mathbf{cap}(\boldsymbol{m}_1, \boldsymbol{m}_2)}{\boldsymbol{\mu}(\boldsymbol{m}_2)} \leq C N^n \mathrm{e}^{-\beta N \Delta_1}.$$

On the hand, for any $A \subset \mathcal{S} \setminus (M_1 \cup M_2)$ Lemma 4.1 implies that

$$\mathbb{P}_{\mu_A}\big[\tau_{M_1 \cup M_2} < \tau_A\big] \geq |\Gamma^n|^{-1}\, \mathrm{e}^{-4\beta\varepsilon(n)(2N+1)} \min_{\boldsymbol{x} \in \Gamma^n \setminus \boldsymbol{B}} \frac{\mathbf{cap}(\boldsymbol{x}, \boldsymbol{M})}{\boldsymbol{\mu}(\boldsymbol{x})},$$

where $\boldsymbol{M} = \{\boldsymbol{m}_1, \boldsymbol{m}_2\}$. For any $\boldsymbol{x} \in \Gamma^n \setminus \boldsymbol{M}$ a lower bound on the mesoscopic capacity $\mathbf{cap}(\boldsymbol{x}, \boldsymbol{M})$ follows by standard comparison with the capacity $\mathbf{cap}_\gamma(\boldsymbol{x}, \boldsymbol{M})$ of a one-dimensional path connecting $\boldsymbol{x}$ with $\boldsymbol{M}$ explicitly computable. For $\boldsymbol{x} \notin \boldsymbol{M}$ there exists a cycle free mesoscopic path $\boldsymbol{\gamma} = (\gamma_0, \dots, \gamma_k)$ in Γ^n such that $\gamma_0 = \boldsymbol{x}$, $\gamma_k \in \boldsymbol{M}$, $\boldsymbol{r}(\gamma_i, \gamma_{i+1}) > 0$ for all $i \in \{0, \dots, k-1\}$ and $\Phi(\boldsymbol{x}, \boldsymbol{M}) = \max_{i \in \{0,\dots,k\}} F(\gamma_i)$, where $\Phi(\boldsymbol{x}, \boldsymbol{M})$ is the communication height as defined in (1.24). In particular, by [5, Proposition 3.1], there exists $C < \infty$ such that for any $N \geq N_0(h) \vee N_1(h)$

$$\frac{\boldsymbol{\mu}(\boldsymbol{x})}{\boldsymbol{\mu}(\gamma_i)} \leq C N^n \mathrm{e}^{\beta N \Delta_2}, \qquad \forall\, i \in \{0, \dots, k\}. \tag{4.8}$$

Hence,

$$\frac{\mathbf{cap}(\boldsymbol{x}, \boldsymbol{M})}{\boldsymbol{\mu}(\boldsymbol{x})} \geq \frac{\mathbf{cap}_\gamma(\boldsymbol{x}, \boldsymbol{M})}{\boldsymbol{\mu}(\boldsymbol{x})} = \left(\sum_{i=0}^{k-1} \frac{\boldsymbol{\mu}(\boldsymbol{x})}{\boldsymbol{\mu}(\gamma_i)\, \boldsymbol{r}(\gamma_i, \gamma_{i+1})} \right)^{-1} \geq \frac{\mathrm{e}^{-\beta(2+h_\infty)}}{k C N^{n+1}}\, \mathrm{e}^{-\beta N \Delta_2},$$

where we used in the last step (4.8) and the fact that $\boldsymbol{r}(\boldsymbol{z}, \boldsymbol{z}') \geq N^{-1} \mathrm{e}^{-2\beta(2+H_\infty)}$ for any $\boldsymbol{z}, \boldsymbol{z}' \in \Gamma^n$ with $\boldsymbol{r}(\boldsymbol{z}, \boldsymbol{z}') > 0$. Since the path $\boldsymbol{\gamma}$ is assumed to be cycle free, its length is bounded by $|\Gamma^n|$, which itself is bounded by N^n. Thus, by combining the estimates above and the assumption that $\Delta_1 - \Delta_2 \geq \theta$, we can absorb the subexponential prefactors. That is, for any $c_1 \in (0, \theta)$ there exists $n_0(\theta, c_1)$ such that for all $n \geq n_0(\theta, c_1)$ the following holds: there exists $N_2 < \infty$ such that for every $N \geq N_0(h) \vee N_1(h) \vee N_2$

$$2\,\frac{\max_{M \in \{M_1,M_2\}} \mathbb{P}_{\mu_M}\big[\tau_{\bigcup_{i=1}^{2} M_i \setminus M} < \tau_M\big]}{\min_{A \subset \mathscr{S}\setminus(M_1 \cup M_2)} \mathbb{P}_{\mu_A}\big[\tau_{M_1 \cup M_2} < \tau_A\big]} \;\leq\; \mathrm{e}^{-\beta c_1 N} \;=:\; \varrho.$$

This completes the proof. □

4.2. **Regularity estimates via coupling arguments.** The main objective in this subsection is to show that Assumption 1.8 is satisfied in the random field Curie-Weiss model.

Proposition 4.2. *Let the assumptions of Proposition 1.20 be satisfied. Then, for any $c_2 \in (0, c_1)$ there exists $n_1 \equiv n_1(c_1, c_2, \beta, h_\infty)$ such that for any $n \geq n_0 \vee n_1$, for any $i \neq j \in \{1,2\}$ and $N \geq N_0(h) \vee N_1(h)$*

$$\mathrm{Var}_{\mu_{M_i}}\left[\frac{\nu_{M_i,M_j}}{\mu_{M_i}}\right] \leq \frac{\eta\,\mu[M_i]}{\mathrm{cap}(M_i, M_j)} \qquad \textit{with} \qquad \eta = \mathrm{e}^{-c_2 \beta N}. \tag{4.9}$$

Moreover, if the external field h taking only finite many discrete values (4.9) holds with $\eta = 0$.

Let us emphasize, that although the bound (4.9) can be in principle deduced from [5, Proposition 6.12], we include a proof of Proposition 4.2 based on a coupling construction. Couplings were first applied in the analysis of the classical Curie-Weiss model [29]. Later, this was adapted in [6, Section 3] to coupling estimates for general spin models allowing for random fields. This approach was simplified and generalized to Potts models in [41]. Here, we give a streamlined presentation of [6] thanks to the simplification of [41] in the setting of the random field Curie-Weiss model.

We are going to construct a coupling $(\sigma(t), \varsigma(t))_{t \in \mathbb{N}_0}$ such that $\sigma(t)$ and $\varsigma(t)$ are two versions of the Glauber dynamic of the random field Curie-Weiss model. Hereby, we choose $\sigma(0) \in \rho^{-1}(\boldsymbol{x})$ and $\varsigma(0) \in \rho^{-1}(\boldsymbol{x})$ having the mesoscopic magnetization $\boldsymbol{x} \in \Gamma^n$. We use that the Glauber dynamic of the Curie-Weiss model defined via (1.22) can be implemented by first choosing a site $i \in \{1, \ldots, N\}$ uniform at random and then flipping the spin at this site i with probability defined by the distribution $\nu_{i,\sigma}$ in the following way

$$\nu_{i,\sigma}[-\sigma_i] := N p(\sigma, \sigma^i) \qquad \text{and} \qquad \nu_{i,\sigma}[+1] + \nu_{i,\sigma}[-1] = 1$$

where $\sigma^i_j = \sigma_j$ for all $j \neq i$ and $\sigma^i_i = -\sigma_i$. Let us note that for any such $\sigma, \varsigma \in \rho^{-1}(\boldsymbol{x})$ and $i, j \in \{1, \ldots, N\}$ such that $\rho(\sigma^i) = \rho(\varsigma^j)$, it holds by using two times the bound (4.4) the estimate

$$\mathrm{e}^{-4\beta\varepsilon(n)}\,\nu_{i_\sigma}[-\sigma_i] \;\leq\; \nu_{j,\varsigma}[-\varsigma_j]. \tag{4.10}$$

The first objective is to couple the probability distributions $\nu_{i,\sigma}$ and $\nu_{j,\varsigma}$ for $\sigma, \varsigma \in \rho^{-1}(\boldsymbol{x})$ with i, j chosen such that $\sigma_i = \varsigma_j$. Taking advantage of the bound (4.10), the coupling can be constructed in such a way that we can decide in advance by tossing a coin whether the same spin values are attained after the coupling step.

The actual construction of the coupling is a modification of the optimal coupling result on finite point spaces introduced in [30, Proposition 4.7]. The constant $\mathrm{e}^{-4\beta\varepsilon(n)}$ from (4.10) will play the role of δ, when we apply the following Lemma 4.3.

Lemma 4.3 (Optimal coupling [41, Lemma 2.3]). *Let $\nu, \nu' \in \mathscr{P}(\{-1,1\})$ and suppose that there exists $\delta \in (0,1)$ such that $\delta\,\nu(s) \leq \nu'(s)$ for $s \in \{-1,1\}$. Then, there exists an optimal coupling (X,X') of ν and ν' with the additional property that for a Bernoulli-δ-distributed random variable V independent of X it holds that*

$$\mathbb{P}\big[X' = s' \mid V = 1, X = s\big] = \mathbb{1}_s(s') \qquad \text{for } s,s' \in \{-1,1\}$$

Therewith, we are able to describe the coupling construction. Let $T > 0$ and $M > 0$ and choose a family $(V_i : i \in \{1,\dots,M\})$ of i.i.d. Bernoulli variables with

$$\mathbb{P}[V_i = 1] = 1 - \mathbb{P}[V_i = 0] = \mathrm{e}^{-4\beta\varepsilon(n)}.$$

The coupling is initialized with $\sigma(0) = \sigma$, $\varsigma(0) = \varsigma$, $M_0 = 0$ and $\xi = 0$.

for $t = 0,1,\dots,T-1$ **do**
  if $\xi = 0$ and $M_t < M$ **then**
    Choose i uniform at random in $\{1,\dots,N\}$ and set $I_t = i$.
    if $\sigma_i(t) = \varsigma_i(t)$ **then**
      Choose $s \in \{-1,1\}$ at random according to $\nu_{i,\sigma}$ and set

$$\sigma_j(t+1) = \begin{cases} \sigma_j(t), & j \neq i \\ s, & j = i \end{cases} \quad \text{and} \quad \varsigma(t+1) = \begin{cases} \varsigma_j(t), & j \neq i \\ s, & j = i \end{cases}.$$

      Set $M_{t+1} = M_t$.
    else
      Let ℓ be such that $i \in \Lambda_\ell$.
      Choose j uniform at random in $\{j \in \Lambda_\ell : \varsigma_j \neq \sigma_j \text{ and } \varsigma_j = \sigma_i\}$.
      Apply Lemma 4.3 to the distributions $\nu_{i,\sigma}$ and $\nu_{j,\varsigma}$, where V_{M_t} decide if both chains maintain the same mesoscopic value.
      Set $M_{t+1} = M_t + 1$
      if $V_{M_t} = 0$ **then**
        Set $\xi = 1$
      end if
    end if
  else
    Use the independent coupling to update $\sigma(t)$ and $\varsigma(t)$
  end if
end for

Lemma 4.4 (Coupling property). *The joint probability measure $\mathbb{P}_{\sigma,\varsigma}$ of the processes $((\sigma(t)),(\varsigma(t)),(V_t) : t \in \{1,\dots,T\})$ obtained from the construction above is a coupling of two versions of the random field Curie-Weiss model started in σ and ς, respectively.*

Proof. As soon as $\xi = 1$ or $M_t \geq M$ for some $t < T$, the both chains evolve independently and are coupled. For $\xi = 0$ and $M_t < M$, we have the by construction i is chosen uniform at random among $\{1, \ldots, N\}$. Then, in the case $\sigma_i = \varsigma_i$ it holds $\nu_{i,\sigma} = \nu_{i,\varsigma}$ and in the other case Lemma 4.3 ensures the coupling property. □

The coupling construction ensures that, once $\varsigma(t)$ and $\sigma(t)$ merged, they evolve together till time T. Hence, we would like to call the event $\{\sigma(t) = \varsigma(t)\}$ a successful coupling. Since conditioning on this event distorts in general the statistical properties of the paths ς, we will introduce two independent subevents which are sufficient to ensure a merging of the processes in time T.

Lemma 4.5. *For any value T and M define the following two events.*

(1) *The event that all Bernoulli variables V_i are ones, i.e.*
$$\mathscr{A} := \{V_i = 1 : i \in \{0, \ldots, M-1\}\}.$$
(2) *The stopping time $\mathfrak{t}_i$ is the first time the i-th spin flips and $\mathfrak{t}$ is first time all coordinates of σ have been flipped, that is*
$$\mathfrak{t}_i = \inf\{t \geq 0 : \sigma(t+1) = -\sigma(0)\} \qquad \text{and} \qquad \mathfrak{t} := \max_{i \in \{1,\ldots,N\}} \mathfrak{t}_i.$$
Therewith, the random variable
$$\mathscr{N} := \sum_{i=1}^{N} \sum_{t=0}^{\mathfrak{t}_i} \mathbb{1}_{I_t = i}.$$
represents the total number of flipping attempts until time $\mathfrak{t}$. The event $\mathscr{B}$, only depending on $\{\sigma(t) : t \in \{0, \ldots, T\}\}$, is defined for any $B \subset \mathscr{S}$ by
$$\mathscr{B} := \{\mathfrak{t} \leq \tau_B\} \cap \{\mathscr{N} \leq M\}. \tag{4.11}$$

Then, it holds that
$$\mathscr{A} \cap \mathscr{B} \subset \{\sigma(\mathfrak{t}) = \varsigma(\mathfrak{t})\}.$$

Proof. The event $\mathscr{B}$ ensures that $\sigma(t)$ has not reached the set B and all its spins have flipped once. By the event $\mathscr{A}$ each flipping aligns one more spin with $\varsigma(t)$ and hence we have $\sigma(\mathfrak{t}) = \varsigma(\mathfrak{t})$. □

By construction, we have
$$\mathbb{P}_{\varsigma,\sigma}[\mathscr{A}] \leq e^{-4\beta\varepsilon(n)M}. \tag{4.12}$$
which is exponentially small in M. Therefore, we have to ensure that the event $\mathscr{B}$ is large. For the first subevent in its definition (4.11), this ensured in a metastable situation. For the second, this is provided for T sufficiently large. For the third one, we use the observation that the rate of spin changes is uniformly bounded from below thanks to the boundedness assumption (1.21). This is for any $\sigma \in \mathscr{S}$ and $i \in \{1, \ldots, N\}$ it holds
$$\nu_{i,\sigma}[-\sigma_i] \geq \exp\bigl(-2\beta(1 + h_\infty)\bigr)$$

Lemma 4.6. *Let $s > \alpha^{-1} := \exp(2\beta(1+h_\infty))$ and set $M = c_3 N$. Then, it holds that*

$$\mathbb{P}_\sigma[\mathcal{N} > M] \;\le\; \mathrm{e}^{-I_\alpha^{\mathrm{nBer}}(s-1)N},$$

were I_α^{nBer} is strictly convex the rate functional of the negative Bernoulli distribution with parameter α given for $s > 1$ by

$$I_\alpha^{\mathrm{nBer}}(s-1) \;:=\; (s-1)\ln\frac{s-1}{s(1-\alpha)} - \ln(\alpha s) \;\ge\; 0. \tag{4.13}$$

In particular $I_\alpha^{\mathrm{nBer}}(s-1) > 0$ for all $s > \alpha^{-1}$.

Proof. The bound implies that if a site is chosen among $\{1,\dots,N\}$, it is flipped with probability at least α. Therefore, let $(\omega(t) : t \in \{0,\dots,T\})$ be a family of independent $\mathrm{Ber}(\alpha)$-distributed random variables and define the negative binomial distributed random variable $\mathcal{R}$ with parameters N and $1-\alpha$ by

$$\mathcal{R} \;:=\; \inf\Big\{s \ge 1 \,:\, \sum_{t=1}^{s} \omega(t) = N\Big\} - N.$$

Then, by using a straightforward coupling argument (see [41, Lemma 2.6]), we obtain that $\mathcal{N} \le \mathcal{R} + N$. Further, by standard large deviation estimates follows

$$\mathbb{P}_\sigma[\mathcal{N} > sN] \;\le\; \mathbb{P}[\mathcal{R} > (s-1)N] \;\le\; \mathrm{e}^{-N I_\alpha^{\mathrm{nBer}}(s-1)}. \tag{4.14}$$

Hereby, I_α^{nBer} is given as the Legendre-Fenchel dual of the log-Moment generating function of the negative Bernoulli distribution, that is $\log\big(\alpha/(1-(1-\alpha)e^t)\big)$. The rate function I_α^{nBer} is strictly convex, since $\partial_s^2 I_\alpha^{\mathrm{nBer}}(s-1) = \frac{1}{s(s-1)}$ and has its unique minimum in $s = \alpha^{-1}$. Hence, $I_\alpha^{\mathrm{nBer}}(s-1)$ is strictly positive for $s > \alpha^{-1}$. □

The above construction allows to deduce the following bound on hitting probabilities of preimages of mesoscopic sets.

Lemma 4.7. *For any $n \in \mathbb{N}$ and $\boldsymbol{A}, \boldsymbol{B} \subset \Gamma^n$ disjoint, set $A = \rho^{-1}(\boldsymbol{A})$ and $B = \rho^{-1}(\boldsymbol{B})$. Further, let $\boldsymbol{x} \in \Gamma^n$ and choose $s > \alpha^{-1}$ according to Lemma 4.6. Then,*

$$\mathbb{P}_\varsigma[\tau_B < \tau_A] \;\ge\; \mathrm{e}^{-4\beta\,\varepsilon(n) s N}\Big(\mathbb{P}_\sigma[\tau_B < \tau_A] - \mathrm{e}^{-I_\alpha^{\mathrm{nBer}}(s-1)N}\Big), \qquad \forall\, \sigma, \varsigma \in \rho^{-1}(\boldsymbol{x}) \tag{4.15}$$

where I_α^{nBer} is given in (4.13).

Proof. We are going to use the above coupling construction with involved parameter $T = \infty$ and $M = sN$. For that purpose, consider the following additional event

$$\overline{\mathcal{B}} \;:=\; \big\{\tau_B \le \mathfrak{t}\big\} \cap \{\mathcal{N} \le M\}.$$

Notice that by Lemma 4.5, on the event $\mathcal{A} \cap \mathcal{B}$ we have $\sigma(\mathfrak{t}) = \varsigma(\mathfrak{t})$ and, in particular, $\tau_B^\sigma = \tau_B^\varsigma$. Moreover, on the event $\mathcal{A} \cap \mathcal{B} \cap \{\tau_B^\varsigma < \tau_A^\varsigma\}$, it follows that $\tau_A^\varsigma = \tau_A^\sigma$.

On the event $\mathcal{A} \cap \overline{\mathcal{B}}$, the process $(\varsigma(t) : t \ge 0)$ reaches B before time $\mathfrak{t}$. However, by the coupling construction, we have that $\rho(\sigma(t)) = \rho(\varsigma(t))$ for all $t \le \tau_B$. Since, by assumption the sets A, B are preimages of the mesoscopic sets $\boldsymbol{A}, \boldsymbol{B}$, we conclude

$\tau_B^\sigma = \tau_B^\varsigma$ and on the event $\{\tau_B^\varsigma < \tau_A^\varsigma\}$ the σ-chain can not reach A before time τ_B^σ. Thus, we have

$$\begin{aligned}
\mathbb{P}_\varsigma\big[\tau_B < \tau_A\big] &\geq \mathbb{P}_{\varsigma,\sigma}\big[\tau_B^\varsigma < \tau_A^\varsigma, \mathscr{A} \cap \mathscr{B}\big] + \mathbb{P}_{\varsigma,\sigma}\big[\tau_B^\varsigma < \tau_A^\varsigma, \mathscr{A} \cap \overline{\mathscr{B}}\big] \\
&= \mathbb{P}_{\varsigma,\sigma}\big[\tau_B^\sigma < \tau_A^\sigma, \mathscr{A} \cap \mathscr{B}\big] + \mathbb{P}_{\varsigma,\sigma}\big[\tau_B^\sigma < \tau_A^\sigma, \mathscr{A} \cap \overline{\mathscr{B}}\big] \\
&\geq \mathbb{P}_{\varsigma,\sigma}[\mathscr{A}]\Big(\mathbb{P}_\sigma\big[\tau_B < \tau_A\big] - \mathbb{P}_\sigma\big[\mathscr{N} > M\big]\Big),
\end{aligned}$$

which concludes the statement thanks to the estimates (4.12) and (4.14). □

We are now in the position to apply the above Lemma to the metastable situation of Proposition 4.2 and use the connection of hitting probabilities and the last exit biased distribution in (1.5).

Proof of Proposition 4.2. For an arbitrary $n \in N$ choose $\{\boldsymbol{a}\}, \boldsymbol{B} \subset \Gamma^n$ disjoint and set $A := \rho^{-1}(\boldsymbol{a})$ and $B := \rho^{-1}(\boldsymbol{B})$. Then, Lemma 4.7 implies that

$$e_{A,B}(\sigma) \overset{(1.5)}{=} \mathbb{P}_\sigma[\tau_B < \tau_A] \overset{(4.15)}{\leq} \mathrm{e}^{4\beta\,\varepsilon(n)\,s\,N}\left(\frac{\operatorname{cap}(A,B)}{\mu[A]} + \mathrm{e}^{-I_\alpha^{\mathrm{nBer}}(s-1)N}\right), \qquad \forall\, \sigma \in A.$$

Hence, in view of (1.10) we obtain

$$\begin{aligned}
\operatorname{Var}_{\mu_A}\left[\frac{\nu_{A,B}}{\mu_A}\right] &= \frac{\mu[A]}{\operatorname{cap}(A,B)}\,\mathrm{E}_{\nu_{A,B}}\big[e_{A,B}\big] - 1 \\
&\leq \frac{\mu[A]}{\operatorname{cap}(A,B)}\left(\left(\mathrm{e}^{4\beta\varepsilon(n)sN} - 1\right)\frac{\operatorname{cap}(A,B)}{\mu[A]} + \mathrm{e}^{(4\beta\varepsilon(n)s - I_\alpha^{\mathrm{nBer}}(s-1))N}\right).
\end{aligned}$$

In particular, under the assumptions of Proposition 1.20, we conclude from the estimate above that

$$\begin{aligned}
&\left(\mathrm{e}^{4\beta\varepsilon(n)sN} - 1\right)\frac{\operatorname{cap}(M_1,M_2)}{\mu[M_2]} + \mathrm{e}^{(4\beta\varepsilon(n)s - I_\alpha^{\mathrm{nBer}}(s-1))N} \\
&\qquad\qquad \leq \mathrm{e}^{4\beta\varepsilon(n)I_\alpha^{\mathrm{nBer}}(s-1)N}\left(\mathrm{e}^{-c_1\beta N} + \mathrm{e}^{-I_\alpha^{\mathrm{nBer}}(s-1)N}\right).
\end{aligned}$$

We have to show that the right hand side is smaller $e^{-c_2\beta N}$ as state in (4.9). Therefore, by the explicit definition of the rate function I_α^{nBer} in (4.13), we can choose s large enough such that $I_\alpha^{\mathrm{nBer}}(s-1) \geq \beta\, c_1$ with c_1 as in Proposition 1.20. Then, for any $c_2 \in (0, c_1)$, we find $n_1 = n_1(c_1, c_2, h_\infty)$ such that for all $n > n_1$, it follows $4\varepsilon(n)s = 8h_\infty s/n < c_1 - c_2$ and hence $\eta = e^{-c_2\beta N}$ as stated in (4.9). □

4.3. **Local mixing estimates within metastable sets.** For the Proof of Proposition 4.8, we follow [31] to compare the Poincaré $C_{\mathrm{PI},i}$ in (1.15) and logarithmic Sobolev constant $C_{\mathrm{LSI},i}$ in (1.16) for any $M \in \mathscr{M}$ with the ones of the Bernoulli-Laplace model. Therefore, we have to first compare the variance and entropy. Then, we introduce the Bernoulli-Laplace model, for which we provide its Poincaré and logarithmic Sobolev constant from the literature. Finally, we have to compare the Dirichlet forms to deduce a Poincaré and logarithmic Sobolev constant inside the metastable sets.

Step 1: Comparison of variance and entropy. We compare the variance and entropy with respect to μ_M with the ones with respect to $\bar{\mu}_M$, where by definition $\bar{\mu}_M$ is the uniform measure on M just denoted by π_M. For the comparison of the variance, we use the two sided comparison

$$\overline{H}(\sigma) - \varepsilon N \le H(\sigma) \le \overline{H}(\sigma) + \varepsilon N \tag{4.16}$$

and obtain

$$\mathrm{Var}_{\mu_M}[f] \;=\; \inf_{a\in\mathbb{R}} \mathrm{E}_{\mu_M}\big[(f-a)^2\big] \;\le\; \mathrm{e}^{\beta\varepsilon(n)N}\,\mathrm{Var}_{\pi_M}[f]. \tag{4.17}$$

Similarly, for the entropy we us the fact that $b\log b - b\log a - b + a \ge 0$ for any $a, b > 0$ and essentially the same argument following [26]

$$\mathrm{Ent}_{\mu_M}\big[f^2\big] \;=\; \inf_{a>0} \mathrm{E}_{\mu_M}\big[f^2\log f^2 - f^2\log a - f^2 + a\big] \;\le\; \mathrm{e}^{\beta\varepsilon(n)N}\,\mathrm{Ent}_{\pi_M}\big[f^2\big].$$

Step 2: Poincaré and logarithmic Sobolev constant of the Bernoulli-Laplace model. Now, let us introduce a nearest neighbor exchange Dirichlet form on M by defining the rates for $\sigma, \sigma' \in M$. Let us introduce the spin-exchange configuration $\sigma_i^{j,k} := \sigma_i$ for $i \notin \{j,k\}$ and $\sigma_j^{j,k} := \sigma_k$ as well as $\sigma_k^{j,k} := \sigma_j$. Then, since $M = \rho^{-1}(\boldsymbol{m})$, we have for $\sigma \in M$, that $\sigma^{j,k} \in M$ if and only if $j,k \in \Lambda_\ell$ for some $\ell \in \{1,\dots,n\}$. Hence, for $\sigma, \sigma' \in M$ with $|\sigma - \sigma'|_1 = 4$, we find $\ell \in \{1,\dots,n\}$ and $j,k \in \Lambda_\ell$ such that $\sigma' = \sigma^{j,k}$. Let us denote the according mesoscopic index by $\ell(\sigma,\sigma')$. Therewith, we can define the transition probabilities by

$$\bar{p}_{\mathrm{BL}}(\sigma,\sigma') \;:=\; \begin{cases} \frac{1}{|\Lambda_{\ell(\sigma,\sigma')}|}, & |\sigma - \sigma'|_1 = 4 \\ 0, & |\sigma-\sigma'|_1 > 4\,. \\ 1 - \sum_{\eta\in M\setminus\sigma} \bar{p}_{BL}(\sigma,\eta), & \sigma = \sigma' \end{cases} \tag{4.18}$$

The measure π_M tensorizes $|M| = \prod_{\ell=1}^n \binom{|\Lambda_\ell|}{k_\ell}$ with $k_\ell = \boldsymbol{m}_\ell N$ and $\rho^{-1}(\boldsymbol{m}) = M$. The rates are compatible with the tensorization, since any jump only occurs among two coordinates in Λ_ℓ for some $\ell \in \{1,\dots,n\}$. Hence, if we regard the coordinates of σ such that $\sigma_i = +1$ as particle position, then the chain induced by $\bar{p}_{\mathrm{BL}}$ is an exclusion process of particles in n boxes of size $\{\Lambda_\ell\}_{\ell=1}^n$ such that in each box $\ell \in \{1,\dots,n\}$ the particle number is k_ℓ. This is the product of n Bernoulli-Laplace models and its spectral gap and logarithmic Sobolev constant is well-known. Let us denote by $\overline{\mathscr{E}}_{\mathrm{BL}}$ the Dirichlet form corresponding to the rates $\bar{p}_{\mathrm{BL}}$ defined in (4.18) with reversible measure π_M. Then by the tensorization property of the Poincaré and logarithmic Sobolev constant [21, Lemma 3.2], we can use the results [22] and [28, Theorem 5] and obtain

$$\mathrm{Var}_{\pi_M}[f] \;\le\; \max_{\ell\in\{1,\dots,n\}} \Big\{C_{\mathrm{PI,BL}(|\Lambda_\ell|,k_\ell)}\Big\}\, \overline{\mathscr{E}}_{\mathrm{BL}}(f)$$

$$\mathrm{Ent}_{\pi_M}\big[f^2\big] \;\le\; \max_{\ell\in\{1,\dots,n\}} \Big\{C_{\mathrm{LSI,BL}(|\Lambda_\ell|,k_\ell)}\Big\}\, \overline{\mathscr{E}}_{\mathrm{BL}}(f).$$

where for some universal constant $c_{\mathrm{BL}} > 0$

$$C_{\mathrm{PI,BL}(|\Lambda_\ell|,k_\ell)} := \left(\frac{|\Lambda_\ell|}{k_\ell(|\Lambda_\ell|-k_\ell)}\right)^{-1} \leq \frac{N}{4} \tag{4.19}$$
$$C_{\mathrm{LSI,BL}(|\Lambda_\ell|,k_\ell)} := C_{\mathrm{PI,BL}(|\Lambda_\ell|,k_\ell)}\left(c_{\mathrm{BL}} \log\frac{|\Lambda_\ell|^2}{k_\ell(|\Lambda_\ell|-k_\ell)}\right)^{-1} \leq \frac{N}{8\log 2\, c_{\mathrm{BL}}}.$$

Step 3: Comparison of Dirichlet forms. For the comparison, we note that the rates $\bar{p}_{\mathrm{BL}}(\sigma,\sigma')$ in (4.18) are not absolutely continuous with respect to the rates $p(\sigma,\sigma')$ defined in (1.22). However, we can define the two step Dirichlet form $\mathscr{E}_2$ with rates $p_2(\sigma,\sigma')$ given by

$$p_2(\sigma,\sigma') := \sum_{\sigma''} p(\sigma,\sigma'')p(\sigma'',\sigma').$$

First, we have the following comparison. Since μ is reversible with respect to p, μ is also reversible with respect to p_2. In addition, p and p_2 have the same eigenvectors φ_j and if we denote by $-1 \leq \lambda_j \leq 1$ according eigenvalue of p, then the jth eigenvalue of p_2 is λ_j^2. Hence, we find the bound

$$\mathscr{E}_2(f) = \sum_{j=1}^{|\mathscr{S}_n|}(1-\lambda_j^2)\left|\langle f,\varphi_j\rangle_\mu\right|^2 \leq 2\sum_{j=1}^{|\mathscr{S}_n|}(1-\lambda_j)\left|\langle f,\varphi_j\rangle_\mu\right|^2 = 2\mathscr{E}(f). \tag{4.20}$$

Now, the last step is to find a bound on the ratio of the rates $\pi_M(\sigma)\bar{p}_{\mathrm{BL}}(\sigma,\sigma')$ and $\mu(\sigma)p_2(\sigma,\sigma')$ for $\sigma,\sigma' \in M$. For $\sigma,\sigma' \in M$ with $|\sigma-\sigma'|_1 = 4$, we find σ'' such that $|\sigma-\sigma''|_1 = 2$ and $|\sigma'-\sigma''|_1 = 2$, which allows to obtain the lower bound using the explicit representation of the Hamiltonian (1.20) as well as the boundedness of the external field (1.21)

$$p_2(\sigma,\sigma') \geq p(\sigma,\sigma'')p_2(\sigma'',\sigma) \geq \frac{1}{N^2}\exp\bigl(-4\beta(1+h_\infty)\bigr).$$

This and using the bound (4.16) and the trivial estimate $|\Lambda_\ell| \geq 1$ leads to

$$\frac{\pi_M(\sigma)\bar{p}_{\mathrm{BL}}(\sigma,\sigma')}{\mu(\sigma)p_2(\sigma,\sigma')} \leq N^2\exp\bigl(\beta(\varepsilon(n)N+4+4h_\infty)\bigr), \tag{4.21}$$

which results in a comparison of the Dirichlet form $\bar{\mathscr{E}}_{\mathrm{BL}}$ and $\mathscr{E}_2$ with the same constant.

Proposition 4.8. *Assumption 1.10 holds with constants $C_{\mathrm{PI},\mathscr{M}}$ and $C_{\mathrm{LSI},\mathscr{M}}$ satisfying*

$$\max\Bigl\{C_{\mathrm{PI},\mathscr{M}}, 2\log 2c_{\mathrm{BL}}C_{\mathrm{LSI},\mathscr{M}}\Bigr\} \leq \frac{N^3}{2}\exp\bigl(2\beta(\varepsilon(n)N+2+2h_\infty)\bigr)$$

for some universal $c_{\mathrm{BL}} > 0$.

Proof Proposition 4.8. The conclusion follows by combining the chain of estimates for the variance

$$\operatorname{Var}_{\mu_M}[f] \overset{(4.17)}{\le} \mathrm{e}^{\beta\varepsilon(n)N} \operatorname{Var}_{\pi_M}[f] \overset{(4.19)}{\le} \frac{N}{4}\,\mathrm{e}^{\beta\varepsilon(n)N}\,\overline{\mathscr{E}}_{\mathrm{BL}}(f)$$
$$\overset{(4.21)}{\le} \frac{N^3}{4}\,\mathrm{e}^{2\beta(\varepsilon(n)N+2+2h_\infty)}\,\mathscr{E}_2(f) \overset{(4.20)}{\le} \frac{N^3}{2}\,\mathrm{e}^{2\beta(\varepsilon(n)N+2+2h_\infty)}\,\mathscr{E}(f)$$

and likewise for the entropy. Since the constant $C_{\mathrm{PI},\mathscr{M}}$ and $C_{\mathrm{LSI},\mathscr{M}}$ are convex combinations over all $C_{\mathrm{PI},M}$ and $C_{\mathrm{LSI},M}$ for $M \in \mathscr{M}$ and the final estimate on $C_{\mathrm{PI},M}$ and $C_{\mathrm{LSI},M}$ is independent of M, we obtain the statement. □

APPENDIX A. YOUNG FUNCTIONS

Lemma A.1 (Properties of Young functions). *A function $\Phi : [0,\infty) \to [0,\infty]$ is a Young function if it is convex, $\Phi(0) = \lim_{r\to 0}\Phi(r) = 0$ and $\lim_{r\to\infty}\Phi(r) = \infty$. Then, it holds that*

(i) Φ is non-decreasing;
(ii) its Legendre-Fenchel dual $\Psi : [0,\infty) \to [0,\infty]$ defined by

$$\Psi(r) := \sup_{s\in[0,\infty]} \{sr - \Phi(s)\}$$

is again a Young function;
(iii) the (pseudo)-inverse of Φ, defined by $\Phi^{-1}(t) := \inf\{s \in [0,\infty] : \Phi(s) > t\}$ is concave and non-decreasing.

Proof. (i). By convexity, it holds for any $\alpha \in (0,1)$ that

$$\Phi(\alpha t) = \Phi(\alpha t + (1-\alpha)\cdot 0) \le \alpha\Phi(t) + (1-\alpha\Phi(0) = \alpha\Phi(t).$$

Hence, by using additionally the non-negativity of Φ it follows for any $\alpha \in (0,1)$

$$\Phi(t) \ge \frac{1}{\alpha}\Phi(\alpha t) \ge \Phi(\alpha t).$$

(ii). The convexity of Ψ follows by convex duality for Legendre-Fenchel transform, since Φ is a convex function. Since $\Phi(s) \ge 0$ for all s and at least equality for $s = 0$, it first follows $\Psi(r) \ge 0$ for all r and in particular

$$\Psi(0) = \sup_{s\in[0,\infty]} \{-\Phi(s)\} = 0.$$

Now, from $\lim_{r\to\infty}\Phi(r) = \infty$ and the convexity of Φ follows that, there exists $\kappa > 0$ such that $\Phi(r) \ge \kappa r$ for $r \ge R$. Hence, we get

$$\lim_{r\to 0} \sup_{s\in[0,\infty]} \{sr - \Phi(s)\} \le \lim_{r\to 0} \max\left\{ \sup_{s\in[0,R]} sr, \sup_{s\ge R}\{s(r-\kappa)\} \right\} = 0$$

Similarly, since $\lim_{r\to\infty}\Phi(r) = 0$, it follows that $\Phi(r) \le \varepsilon < \infty$ for all $r \in [0,\delta]$ and hence

$$\lim_{r\to\infty} \sup_{s\in[0,\infty]} \{sr - \Phi(s)\} \ge \lim_{r\to\infty} (\delta r - \varepsilon) = \infty.$$

(iii). The fact, that Φ^{-1} is non-decreasing follows immediately from its definition and that Φ is non-decreasing. Now, let $u, v \in \{\Phi(s) : s \in \mathbb{R}, \Phi(s) < \infty\}$. Then, by convexity of Φ follows for $\alpha \in (0,1)$ and $\beta = 1-\alpha$

$$\Phi\big(\alpha\Phi^{-1}(u) + \beta\Phi^{-1}(v)\big) \le \alpha\Phi\big(\Phi^{-1}(u)\big) + \beta\Phi\big(\Phi^{-1}(v)\big) = \alpha u + \beta v,$$

where we used that Φ is continuous on its finite support, since it is convex. Since Φ^{-1} is non-decreasing, the inequality is preserved after applying it

$$\Phi^{-1}\Big(\Phi\big(\alpha\Phi^{-1}(u) + \beta\Phi^{-1}(v)\big)\Big) \le \Phi^{-1}(\alpha u + \beta v).$$

Now by noting

$$\Phi^{-1}(\Phi(x)) = \inf\{s : \Phi(s) > \Phi(x)\} \ge x,$$

if follows that Φ^{-1} is concave on the finite range of Φ. If this range is finite, then Φ^{-1} gets extended continuously as a constant and hence still concave. □

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References

[1] C. Ané, S. Blachère, D. Chafaï, P. Fougères, I. Gentil, F. Malrieu, C. Roberto, and G. Scheffer. *Sur les inégalités de Sobolev logarithmiques*, volume 10 of *Panoramas et Synthèses*. Société Mathématique de France, Paris, 2000. With a preface by D. Bakry and M. Ledoux.

[2] D. Bakry, I. Gentil, and M. Ledoux. *Analysis and geometry of Markov diffusion operators*, volume 348 of *Grundlehren der Mathematischen Wissenschaften*. Springer, Cham, 2014.

[3] J. Beltrán and C. Landim. A martingale approach to metastability. *Probab. Theory Related Fields*, 161(1-2):267–307, 2015.

[4] N. Berglund and S. Dutercq. The Eyring-Kramers law for Markovian jump processes with symmetries. *J. Theoret. Probab.*, 29(4):1240–1279, 2016.

[5] A. Bianchi, A. Bovier, and D. Ioffe. Sharp asymptotics for metastability in the random field Curie-Weiss model. *Electron. J. Probab.*, 14:no. 53, 1541–1603, 2009.

[6] A. Bianchi, A. Bovier, and D. Ioffe. Pointwise estimates and exponential laws in metastable systems via coupling methods. *Ann. Probab.*, 40(1):339–371, 2012.

[7] A. Bianchi and A. Gaudillière. Metastable states, quasi-stationary distributions and soft measures. *Stochastic Process. Appl.*, 126(6):1622–1680, 2016.

[8] S. G. Bobkov and F. Götze. Exponential integrability and transportation cost related to logarithmic Sobolev inequalities. *J. Funct. Anal.*, 163(1):1–28, 1999.

[9] S. G. Bobkov and P. Tetali. Modified logarithmic Sobolev inequalities in discrete settings. *J. Theoret. Probab.*, 19(2):289–336, 2006.

[10] A. Bovier. Metastability: a potential theoretic approach. In *International Congress of Mathematicians. Vol. III*, pages 499–518. Eur. Math. Soc., Zürich, 2006.

[11] A. Bovier and F. den Hollander. *Metastability*, volume 351 of *Grundlehren der Mathematischen Wissenschaften*. Springer, Cham, 2015. A potential-theoretic approach.

[12] A. Bovier, M. Eckhoff, V. Gayrard, and M. Klein. Metastability in stochastic dynamics of disordered mean-field models. *Probab. Theory Related Fields*, 119(1):99–161, 2001.

[13] A. Bovier, M. Eckhoff, V. Gayrard, and M. Klein. Metastability and low lying spectra in reversible Markov chains. *Comm. Math. Phys.*, 228(2):219–255, 2002.

[14] A. Bovier, M. Eckhoff, V. Gayrard, and M. Klein. Metastability in reversible diffusion processes. I. Sharp asymptotics for capacities and exit times. *J. Eur. Math. Soc. (JEMS)*, 6(4):399–424, 2004.

[15] A. Bovier, V. Gayrard, and M. Klein. Metastability in reversible diffusion processes. II. Precise asymptotics for small eigenvalues. *J. Eur. Math. Soc. (JEMS)*, 7(1):69–99, 2005.

[16] A. Bovier and F. Manzo. Metastability in Glauber dynamics in the low-temperature limit: beyond exponential asymptotics. *J. Statist. Phys.*, 107(3-4):757–779, 2002.

[17] C. J. Burke and M. Rosenblatt. A Markovian function of a Markov chain. *Ann. Math. Statist.*, 29:1112–1122, 1958.

[18] M. Cassandro, A. Galves, E. Olivieri, and M. E. Vares. Metastable behavior of stochastic dynamics: a pathwise approach. *J. Statist. Phys.*, 35(5-6):603–634, 1984.

[19] J. Cheeger. A lower bound for the smallest eigenvalue of the Laplacian. In *Problems in analysis*, pages 195–199. Princeton Univ. Press, Princeton, N. J., 1970.

[20] M.-F. Chen. *Eigenvalues, inequalities, and ergodic theory*. Probability and its Applications (New York). Springer-Verlag London, Ltd., London, 2005.

[21] P. Diaconis and L. Saloff-Coste. Logarithmic Sobolev inequalities for finite Markov chains. *Ann. Appl. Probab.*, 6(3):695–750, 1996.

[22] P. Diaconis and M. Shahshahani. Time to reach stationarity in the Bernoulli-Laplace diffusion model. *SIAM J. Math. Anal.*, 18(1):208–218, 1987.

[23] M. I. Freidlin and A. D. Wentzell. *Random perturbations of dynamical systems*, volume 260 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, New York, 2nd edition, 1998.

[24] B. Helffer, M. Klein, and F. Nier. Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach. *Mat. Contemp.*, 26:41–85, 2004.

[25] Y. Higuchi and N. Yoshida. Analytic conditions and phase transition for ising models, 1995. Lecture Notes in Japanese.

[26] R. Holley and D. Stroock. Logarithmic Sobolev inequalities and stochastic Ising models. *J. Statist. Phys.*, 46(5-6):1159–1194, 1987.

[27] G. F. Lawler and A. D. Sokal. Bounds on the L^2 spectrum for Markov chains and Markov processes: a generalization of Cheeger's inequality. *Trans. Amer. Math. Soc.*, 309(2):557–580, 1988.

[28] T.-Y. Lee and H.-T. Yau. Logarithmic Sobolev inequality for some models of random walks. *Ann. Probab.*, 26(4):1855–1873, 1998.

[29] D. A. Levin, M. J. Luczak, and Y. Peres. Glauber dynamics for the mean-field Ising model: cut-off, critical power law, and metastability. *Probab. Theory Related Fields*, 146(1-2):223–265, 2010.

[30] D. A. Levin, Y. Peres, and E. L. Wilmer. *Markov chains and mixing times*. American Mathematical Society, Providence, RI, 2009. With a chapter by James G. Propp and David B. Wilson.

[31] N. Madras and Z. Zheng. On the swapping algorithm. *Random Structures Algorithms*, 22(1):66–97, 2003.

[32] P. Mathieu and P. Picco. Metastability and convergence to equilibrium for the random field Curie-Weiss model. *J. Statist. Phys.*, 91(3-4):679–732, 1998.

[33] V. G. Maz′ja. Certain integral inequalities for functions of several variables. In *Problems of mathematical analysis, No. 3: Integral and differential operators, Differential equations (Russian)*, pages 33–68. Izdat. Leningrad. Univ., Leningrad, 1972.

[34] V. G. Maz′ya. *Sobolev spaces with applications to elliptic partial differential equations*, volume 342 of *Grundlehren der Mathematischen Wissenschaften*. Springer, Heidelberg, 2011.

[35] G. Menz and A. Schlichting. Poincaré and logarithmic Sobolev inequalities by decomposition of the energy landscape. *Ann. Probab.*, 42(5):1809–1884, 2014.

[36] L. Miclo. An example of application of discrete Hardy's inequalities. *Markov Process. Related Fields*, 5(3):319–330, 1999.
[37] B. Muckenhoupt. Hardy's inequality with weights. *Studia Math.*, 44:31–38, 1972. Collection of articles honoring the completion by Antoni Zygmund of 50 years of scientific activity, I.
[38] E. Olivieri and M. E. Vares. *Large deviations and metastability*, volume 100 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2005.
[39] M. M. Rao and Z. D. Ren. *Applications of Orlicz spaces*, volume 250 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 2002.
[40] A. Schlichting. *The Eyring-Kramers formula for Poincaré and logarithmic Sobolev inequalities*. PhD thesis, Universität Leipzig, 2012.
[41] M. Slowik. *Contributions to the potential theoretic approach to metastability with applications to the random field Curie-Weiss-Potts model*. PhD thesis, Technische Universität Berlin, 2012.
[42] M. Slowik. A note on variational representations of capacities for reversible and non-reversible Markov chains. available on request, 2012.
[43] M. Sugiura. Metastable behaviors of diffusion processes with small parameter. *J. Math. Soc. Japan*, 47(4):755–788, 1995.

INSTITUT FÜR ANGEWANDTE MATHEMATIK, UNIVERSITÄT BONN
Current address: Endenicher Allee 60, 53115 Bonn, Germany
E-mail address: `schlichting@iam.uni-bonn.de`

TECHNISCHE UNIVERSITÄT BERLIN
Current address: Strasse des 17. Juni 136, 10623 Berlin
E-mail address: `slowik@math.tu-berlin.de`