Norm estimations of the modified Teodorescu transform with application to a multidimensional equation of Airy type

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Abstract. We study versions of a generalized Teodorescu transform. In the 2-dimensional case we can describe the asymptotic behaviour by the help of modified Bessel functions. In 3-dimensional case we only have an upper estimate. Such estimates are necessary to prove the convergence of a semi-discretization method for a higher-dimensional analogue of an equation of Airy's type.

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INTRODUCTION

In the classical 2-dimensional Vekua theory the so-called T-operator takes an essential part. This operator is nothing else than a 2-dimensional weak singular integral operator over a domain in the complex plane, which is right inverse to the Cauchy-Riemann operator. It reads as follows:

$$(T_G u)(z) = -\frac{1}{2\pi\mathbf{i}} \int_G \frac{u(t)}{t-z} d\xi d\eta$$

where $t = \xi + \mathbf{i}\eta$ and G is a domain in $\mathbb{C}$. There exists comprehensive treatises on mapping properties and their applications in complex analysis. An excellent reference is the classical book by Ilja N. Vekua: *"Generalized analytic functions* " (1959) (cf. [11]). It isn't exaggerated to say, that the whole theory of complex partial differential equations is based on operators of such type.

The aim of this article is to study estimates of the generalizations of the T-operator, which is called nowadays *Teodorescu type transforms*. One of the first generalizations one can find in [9]. During the last decades the second author together with K. Gürlebeck developed an operator calculus (cf. [4], [5], [3]), where above all multidimensional generalizations and modifications of the Teodorescu transform were studied. In this way we introduce for $n = 2$ the generalized Teodorescu transform: Let G be a bounded domain in $\mathbb{C}$ and u a continuous function

$$(T_\alpha u)(z) = -\frac{1}{\pi} \int_G \frac{e^{\alpha(t-z)}}{\overline{t-z}} u(t) d\xi d\eta \quad (\alpha \in \mathbb{C}),$$

which is the right-invers to the generalized Dirac operator $D_\alpha = \frac{\partial}{\partial z} + \alpha$. For $n \geq 3$ we define:

$$(T_{\pm i\alpha} u)(x) \quad := \quad -\left(\frac{i\alpha}{2\pi}\right)^{n/2} \int_G |x-y|^{1-n/2} \left[K_{n/2}(\alpha|x-y|)\frac{x-y}{|x-y|} - K_{n/2-1}(\alpha|x-y|)\right] u(y)dy,$$

where

$$K_p(t) := \frac{\pi i}{2} exp((pi\pi)/2) H_p^{(1)}(it)$$

is the modified Bessel function of the second kind. Moreover

$$H_p^{(1)}(t) := J_p(t) + iN_p(t)$$

denotes the Hankel functions, which can be expressed through the Bessel functions of the first and the second kind. In particulary for $p \notin Z$ it holds

$$N_p(t) := \frac{\cos \pi p}{\sin \pi p} J_p(t) - \frac{1}{\sin \pi p} J_{-p}(t).$$

For more information we refer to [2], [1]. These types of Teodorescu transforms play an important role in the solution of time-discretised initial-boundary value problems. Aim of this contribution is to show estimates of the Teodorescu transform with respect to the number α. We apply that for the treatment of a multidimensional analogue of Airy's equation, that can be seen to be the linearization of a corresponding multidimensional version of Korteweg – De Vries equation.

ESTIMATES OF THE KERNEL OF THE TEODORESCU TRANSFORM IN $\mathbb{C}$ ON A DISK

We consider the fundamental solution $E_\alpha(z) = \frac{e^{-\alpha z}}{\bar{z}}$ of the generalized Dirac operator $D_\alpha = \frac{\partial}{\partial z} + \alpha$ where $\alpha \in \mathbb{C}$ Put $z = re^{i\varphi}$ and $\alpha = |\alpha| e^{i \arg \alpha}$.Then we get for the absolute value of the kernel function:

$$|E_\alpha(z)| = \frac{1}{r} |e^{-|\alpha| e^{i \arg \alpha} e^{i\varphi}}| = \frac{e^{-|\alpha| r \cos(\varphi + \arg \alpha)}}{r}$$

Now we compute

$$\begin{aligned}
\int_0^{2\pi} |E_\alpha(re^{i\varphi})| r d\varphi &= \int_0^{2\pi} e^{-r|\alpha| \cos(\varphi + \arg \alpha)} d\varphi \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j r^j |\alpha|^j}{j!} \int_0^{2\pi} \cos^j (\varphi + \arg \alpha) d\varphi \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j r^j |\alpha|^j}{j!} \int_0^{2\pi} \cos^j \varphi d\varphi
\end{aligned}$$

The integral over odd powers of of the cosine is zero and for even powers one obtains :

$$\int_0^{2\pi} |E_\alpha(re^{i\varphi})| r d\varphi = \sum_{k=0}^{\infty} \frac{r^{2k} |\alpha|^{2k}}{(2k)!} \cdot \frac{2\pi (2k)!}{2^{2k} (k!)^2} = 2\pi \sum_{k=0}^{\infty} \frac{r^{2k} |\alpha|^{2k}}{2^{2k} (k!)^2} = 2\pi I_0(|\alpha| r)$$

where I_v is th modified Bessel function of the first kind. The integral over the disk with radius r_0 yields

$$\begin{aligned}
\int_{B_{r_0}} |E_\alpha(z)| dx dy &= \int_0^{r_0} \int_0^{2\pi} |E_\alpha(re^{i\varphi})| r d\varphi dr = 2\pi \int_0^{r_0} I_0(|\alpha| r) dr \\
&= 2\pi \sum_{k=0}^{\infty} \frac{|\alpha|^{2k}}{2^{2k} (k!)^2} \int_0^{r_0} r^{2k} dr = 2\pi \sum_{k=0}^{\infty} \frac{|\alpha|^{2k} r_0^{2k+1}}{(2k+1) 2^{2k} (k!)^2} \\
&= 2\pi \sum_{k=0}^{\infty} \frac{|\alpha|^{2k} r_0^{2k+1}}{1 - \frac{1}{2(k+1)} 2^{2k+1} k! (k+1)!}.
\end{aligned}$$

The term $\left(1 - \frac{1}{2(k+1)}\right)$ is upper and lower estimated now by 1 and $\frac{1}{2}$ resuliting in a lower and upper estimate for our integral. In more detail we obtain

$$\frac{2\pi}{|\alpha|} I_1(|\alpha| r_0) \leq \int_{B_{r_0}} |E_\alpha(z)| \, \mathrm{d}x \, \mathrm{d}y \leq \frac{4\pi}{|\alpha|} I_1(|\alpha| r_0).$$

Over a disc annulus with the radii r_1 and r_2 we get after very fine etimations at the end (cf. [8])

$$2\pi \ln \frac{r_1}{r_0} + \frac{\pi}{|\alpha|} C_\alpha(r_0, r_1) \leq \int_{B_{r_0, r_1}} |E_\alpha(z)| \, \mathrm{d}x \, \mathrm{d}y \leq 2\pi \ln \frac{r_1}{r_0} + \frac{2\pi}{|\alpha|} C_\alpha(r_0, r_1),$$

where

$$C_\alpha(r_0,r_1) := \left(\frac{I_1(2|\alpha|r_1)}{r_1} - \frac{I_1(2|\alpha|r_0)}{r_0}\right).$$

Note that $C_\alpha(r_0,r_1) \geq 0$, because of the strictly monotony of $\frac{I_1(2|\alpha|r)}{r}$ in r.

Further an asymptotic bound for big $|\alpha|$ or big r_0 on the half circle $K_{\arg\alpha,r_0}$ opened arround the argument of α with radius r_0 can be found

$$\int_{K_{\arg\alpha,r_0}} |E_\alpha(z)|\,dx\,dy \sim \frac{2}{\pi|\alpha|}(\ln(2|\alpha|r_0)+\gamma),$$

where γ denotes Euler-Mascheroni constant (cf. [8]).

ESTIMATES OF THE TEODORESCU TRANSFORM IN HIGHER DIMENSIONS

Let a be a positive number and $f_a(t) := t^{1-n/2}K_{n/2}(at)$ as well as $g_a(t) := t^{1-n/2}K_{n/2-1}(at)$. Because of $n \geq 2$ follows that $f_a(t)$ and $g_a(t)$ are decreasing functions. Further it holds

$$\begin{aligned}
|T_a 1| &= \left|\int_G -(\frac{a}{2\pi})^{n/2}[f_a(|\mathbf{x}|)\omega - g_a(|\mathbf{x}|)]dx\right| \\
&\leq \left(\frac{a}{2\pi}\right)^{n/2}\int_G [f_a(|\mathbf{x}|)+g_a(|\mathbf{x}|)]dx \quad \text{because } |\omega| = 0 \\
&= \left(\frac{a_0}{2\pi}\right)^{n/2}\left(\int_{G\cap B}[f_a(|\mathbf{x}|)+g_a(|\mathbf{x}|)]dx + \int_{G\backslash B}[f_a(|\mathbf{x}|)+g_a(|\mathbf{x}|)]dx\right).
\end{aligned}$$

Let B be a ball in $\mathbb{R}^n$ with the same measure of G and say radius R. First an estimate of the second integral over $G\backslash B$ is obtained. Due to $\mathbf{x} \in (G\backslash B)$ follows $|\mathbf{x}| > R$ and therewith

$$f_a(|\mathbf{x}|) \leq f_a(R) \quad \text{and} \quad g_a(|\mathbf{x}|) \leq g_a(R)$$

due to the monotinicity of f_a and g_a. Therewith it yields

$$\begin{aligned}
\int_{G\backslash B}[f_a(|\mathbf{x}|)+g_a(|\mathbf{x}|)]dx &\leq \int_{G\backslash B}[f_a(R)+g_a(R)]dx \\
&= [f_a(R)+g_a(R)]\int_{G\backslash B}dx = \int_{B\backslash G}[f_a(R)+g_a(R)]dx \qquad \text{because :} \quad mes(B) = mes(G).
\end{aligned}$$

Remarking that in $B\backslash G$ follows $|\mathbf{x}| < R$ and therefore $f_a(|\mathbf{x}|) \geq f_a(R)$ as well as $g_a(|\mathbf{x}|) \geq g_a(R)$, further it holds

$$\int_{B\backslash G}[f_a(R)+g_a(R)]dx \leq \int_{B\backslash G}[f_a(|\mathbf{x}|)+g_a(|\mathbf{x}|)]dx.$$

We now have

$$\begin{aligned}
|T_{a_0}1| &\leq \left(\frac{a_0}{2\pi}\right)^{n/2}\left(\int_{B\cap G}[f_{a_0}(|\mathbf{x}|)+g_{a_0}(|\mathbf{x}|)]dx + \int_{B\backslash G}[f_{a_0}(|\mathbf{x}|)+g_{a_0}(|\mathbf{x}|)]dx\right) \\
&= \left(\frac{a_0}{2\pi}\right)^{n/2}\int_B [f_{a_0}(|\mathbf{x}|)+g_{a_0}(|\mathbf{x}|)]dx \\
&= \left(\frac{a_0}{2\pi}\right)^{n/2} n\nu_n\left(\frac{2^{n/2-1}}{a_0^{n/2}+1}\Gamma(n/2) - \frac{R^{n/2}}{a_0}K_{n/2}(a_0R) + R^{n/2+1}K_{n/2}(a_0R) + a_0R\frac{2^{n/2-1}}{a_0^{n/2}+1}\Gamma(n/2)\right) \\
&= \left(\frac{a_0}{2\pi}\right)^{n/2} n\nu_n\left(R - \frac{1}{a_0}\right)\left[\frac{2^{n/2-1}}{a_0^{n/2}}\Gamma(n/2) + R^{n/2}K_{n/2}(a_0R)\right],
\end{aligned}$$

where $\nu_n \in \mathbb{R}$ only depends on n. In [7] one can find the estimations

$$K_\alpha(x) \leq C\frac{e^{-x}}{\sqrt{x}} \quad \text{for} \quad x > 1.$$

For small x and $\nu > 0$ we have always

$$|K_\alpha(x)| \leq C\frac{1}{x^\alpha}.$$

Now one can easily see that for $|\alpha| \to \infty$ the value $|T_{a_0}1|$ tends to zero. For further estimations see also [1]. We have to thank Tran Minh Hoang (TU Hanoi), who helps to confirm this result.

A GENERALIZED AIRY'S TYPE EQUATION

In one dimension it is well-known that Airy's equation

$$\partial_t u = -u_{xxx} + f$$

is a linearization of the famous Korteweg–De Vries equation. A possible multidimensional generalisation is given by

$$\partial_t u = -D^3 u + f.$$

In this equation for u the quaternion-valued function f is given as data and D is the well-known mass-less Dirac operator defined by

$$D^3 := e_1\partial_1 + e_2\partial_2 + e_3\partial_3,$$

where e_1, e_2, e_3 are together with $e_0 = 1$ the quaternionic units. Additionally, initial-values $u(0,\cdot) = u_0(\cdot)$ and boundary values g are known.

Set $T = n\tau$, where τ is the meshwidth of the method. We further use the abbreviations

$$\mathbf{u}_k := u(k\tau,\cdot) \quad \text{and} \quad f_k = \frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} f(t,x)dt \quad \text{as well as} \quad g_k(\cdot) = g(k\tau,\cdot).$$

Now we substitute the partial derivative $(\partial_t \mathbf{u})(k\tau,\cdot)$ by the finite forward differences

$$\frac{\mathbf{u}_{k+1} - \mathbf{u}_k}{\tau} \quad (k = 0,1,...,,n-1).$$

The generalized Airy's equation should be approximated by

$$u_{k+1} - u_k = \tau D^3 u_{k+1} + f_k.$$

Setting $\tau =: a^3$ and $b = a^{-1}$ we obtain

$$(I - a^2 D^3 u_{k+1} = (I - aD)(I + aD + a^2D^2)u_{k+1} = u_k + f_k =: F_k.$$

and

$$(I + aD + a^2D^2)u_{k+1} = T_{-b}bF_k + \phi_a \quad (\phi_a \in \ker(D+a))$$

After factorization with the well defined complex number c we gain

$$u_{k+1} = T_c T_{\overline{c}} T_{-b} b|c|^2 F_k + T_c T_{\overline{c}} |c|^2 \phi_{-b} + T_c c \phi_{\overline{c}} + \phi_c.$$

The quaternionic valued functions belong to kerD_c or respectivley to ker$D_{\overline{c}}$. Using results from [6] we get the representation

$$u_{k+1} = T_c \mathscr{Q}_c T_{\overline{c}} T_{-b} b |c|^2 F_k + T_c T_{\overline{c}} |c|^2 \phi_{-b} + \mathscr{H} g_{k+1}$$

where

$$\mathscr{H} g_{k+1} = F_c g_{k+1} + T_c \mathscr{P}_c (D - \overline{c}) H_{k+1} \qquad (k = 0, ..., n-1).$$

Here H_{k+1} is a smooth continuation of g_{k+1} into the domain G. Further, we have $\mathscr{Q}_c = I - \mathscr{P}_c$, where $\mathscr{P}_c$ denotes the generalized Bergman projection.

The proof of the approximation and stability leads to the same methods already used in [6] and makes use of such type of estimations as presented in the sections before. It is possible to pose an additional (half)-boundary condition for Du.

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