Diploma Thesis

Solvability, approximation and estimates for a class of singular phase field models

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Declaration of Academic Honesty

Hereby I declare that this Diploma Thesis is my own work and does not involve plagiarism or collusion. I also declare that for the material contained in this Diploma Thesis only the listed resources and tools are used.

Freiberg, October 1, 2008 André Schlichting

To my grandfather

A great teacher in mental arithmetic and life

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Chapter 1

Introduction

1. Overview

The topic of this thesis is the phenomena of phase transition. We consider binary systems of first and second order. The mathematical background was developed by Ginzburg and Landau in the 50's and is therefore called *Ginzburg-Landau-theory* [15]. Newer literature is from Fabrizio [13] and [12].

In a first order phase transition phenomena, as in the sollid-liquid or liquid-vapour phase change, the phase transition occurs at a critical temperature ϑ_c, called ϑ_c. If the absolute temperature ϑ in the body is strictly greater than the critical temperature, then the minimum of the energy potential is attained in one of the pure phases, while if $\vartheta < \vartheta_c$ the minimum is attained in the other phase. In the case when $\vartheta = \vartheta_c$ the energy potential has two minima attained for the two phases, that is the phase change may occur. On the other hand, in the case of second order phase transitions, the system behaves differently provided ϑ is greater or less the critical temperature ϑ_c. Indeed, for high temperatures the energy potential has only one minimum, while for $\vartheta < \vartheta_c$ two minima are attained with the same values. This second behaviour is characteristic, for instance, of solid-solid phase transitions, ferromagnetism and superconductivity.

We will investigate a model describing these phenomena by use of phase-field theories, in terms of the temperature ϑ and a phase parameter χ, which includes the effects of micro-motions and micro-forces responsible for the phase transition. Indeed, it is known that the macroscopic event of phase transition is caused by changes occurring at a microscopic level in the (atomic and/or crystal) structure of the system. Theses changes are the effects of micro-forces and micro-motions, which have to be included in the balance of the energy of the whole system, even if we are providing a macroscopic description of the phenomena.We follow the suggestion of Frémond [14], who proposed a new model describing phase transitions by use of a generalisation of the principle of virtual power including the effects of micro-motions and micro-forces. For a more detailed derivation of the model we confer to [7]. A similar theory was developed by Gurtin [16] for Ginzburg-Landau and Cahn-Hilliard equations, but in the isothermal case.

We combine the theory of micro-motions with a model based on a *reduced energy balance* equation. We mainly refer to [5]. The main advantage of the model treated in the thesis is that, once the problem is solved in a suitable sense, we can obtain directly the positivity of the temperature, mainly caused by the presence of a logarithmic nonlinearity in the resulting system of partial differential equations. Therefore no maximum principle arguments have to be applied, which are difficult to set in a number of interesting situations.

2. Structure

The thesis is structured in the following way: The next Chapter 2 explains in a short way the physical background and derives the model. The mathematical explanation is stated in the Chapter 3, where first the continuous problem is stated in detail. Afterwards we derive from that the discrete scheme, for which we prove an existence and uniqueness statement. At this point we show the convergence to the continuous problem. Further we extend the result of [3] by the continuous dependence on the data, wherefor we reproduce the boundedness result already performed in the mentioned paper.

The Chapter 4 provides important mathematical tools, which will be used for the proofs performed in the afterwards chapters. The first section provides a brief introduction to monotone operators. We provide only the theory, which is needed for defining the *Yosida approximation*, which will be used for regularising the logarithm in the existence proof. The next section dealing with subdifferential mappings is closely connected to monotone operators. The only objective here is to state a general chain rule with respect to time, which will be the most important instrument in the boundedness proof. In the section on time approximation is all notation collected, which is later needed to perform the limit to the continuous problem, especially in two lemmata we proof already some technical partial convergence results. After this follows again an approximation section to handle the singularities of the right hand side. Finally the last sections introduces the harmonic extension, with which we can handle the non homogeneous Dirichlet boundary conditions.

In the following Chapters 5 up to 8 are the theorems of the main results performed. First in the Chapter 5 the maximum principle for the phase parameter χ and uniqueness, existence of our system are shown. The first two hypotheses can be derived from monotonicity arguments. The existence proof of the phase parameter uses the theory of monotone operators. However a regularization procedure is applied for the temperature.

The Chapter 6 proves the convergence to a continuous solution. Therefore some a priori estimates are obtained. From these estimtates we can show weak and weak star convergences in suitable spaces. Finally with the help of compact embedding arguments we can conclude the proof.

As already pointed out, the next two chapters handle again with the continuous problem. First, in Chapter 7 we reproduce the proof of the boundedness of the temperature, which is already obtained in [3]. Besides the techniques developed in the according sections of the tools chapter we use a *Moser type technique*: starting from an initial norm estimate for the temperature, we obtain iterative bounds in higher Lebesgue spaces. These norms are summands of a geometric series for which we will show convergence. This will state therefore the boundedness. In addition with easy considerations we will increase the regularity for the temperature and the phase parameter based on the boundedness of ϑ.

In the last Chapter 8 we prove the continuous dependence of the solution on the data. We can only achieve this result by reinforcing the assumptions on the system a little bit. We will have to assume, that the source term is Lipschitz continuous. In addition two important tools will be the Riesz representation operator and a special harmonic extension to handle the non homogene Dirichlet boundary data.

The appendix is split into two parts. In the Appendix A we state some useful theorems from measure theory, which will be especially used for the convergence to a continuous solution. The Appendix B is dedicated to functional analysis. Especially we will widely use embedding theorems for Sobolev spaces, which help us by our several estimates.

Chapter 2

Physical background

We consider a two-phase system pointed already out in [3]. Therefore the system is located in a smooth and bounded domain $\Omega \subset \mathbb{R}^3$ and we watch its evolution during a finite time interval $(0,T)$. We denote by Γ the boundary $\partial\Omega$. The thermomechanical equilibrium of the system is described in terms of a state variable and is governed by the free energy, while the dynamic reflects the presence of a pseudo-potential of dissipation. We do not consider mechanical effects. Thus the variables of the system are just the absolute temperature ϑ and a phase parameter χ, related to the proportion of one phase with respect to the other. In general, χ attains its physical admissible values in a range $[\chi_*, \chi^*]$ (for example $\chi \in [0,1]$) for all time up to T. We will observe that this physical constraint is ensured by the model itself.

1. Evolution equations

1.1. Order parameter χ

The just introduced phase parameter χ can be interpreted also as an order parameter. Thus it describes the change in the order structure of the thermomechanical system. For many materials the order structure below a critical temperature is greater than above. For example if we are thinking of liquid-solid phase transitions, then is the system below the melting temperature in a more ordered state. The phenomena is also characteristic for ferromagnetism. Above the Curie-temperature, the magnetic moments (also called *Weiss domains*) of the system are in a less ordered state and so the material is non magnetic.

Evolution Nevertheless, the parameter χ is a macroscopic parameter, the evolution of χ is governed by the micro-forces and micro-movements responsible for the phase transition at a microscopic level. Thus, the evolution of the order parameter χ can be derived from thermomechanical laws, as a balance equation for micro-forces. The balance conditions are read in the following way

$$B - \operatorname{div} \mathbf{H} = 0 \quad \text{in } \Omega \times (0,T), \quad \mathbf{H} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \times (0,T), \tag{2.1}$$

where $\mathbf{n}$ is the outward normal to the boundary Γ. In addition B can be interpreted as an energy density per units of concentration χ as well as $\mathbf{H}$ represents an energy flux vector. Note that we assume with the homogene Neumann conditions that no external work is carried out to the system.

1.2. Temperature ϑ

For the evolution of the temperature we will use the first and second principle of thermodynamics. We use a rescaled energy balance in which higher order dissipative contributions are neglected by means of the small perturbations assumption.

Evolution We obtain a balance, which could be considered as an "entropy equation", because it describes the evolution of the entropy s of the system in terms of the entropy flux $\mathbf{Q}$ and an external source $R(\vartheta)$, possibly depending on the temperature ϑ and maybe singular. Thus the following equation holds

$$s_t + \operatorname{div} \mathbf{Q} = R(\vartheta) \quad \text{in } \Omega \times (0,T). \tag{2.2}$$

We can demand boundary conditions on the entropy flux. If no flux is assumed through the boundary, then holds $\mathbf{Q} \cdot \mathbf{n} = 0$ on $\Gamma \times (0,T)$. Later we will ask for the value of the temperature on the boundary.

2. Energy functionals

We specify the involved physical quantities with the help of two energy functionals: On the one hand the *free energy* Ψ, depending on the state variables and accounting for the thermomechanical equilibrium of the system. On the other hand the *pseudo-potential of dissipation* Φ, defined for the dissipative variables and responsible for the evolution of the system. For details on these functionals we refer to Moreau [21].

2.1. Free energy

Our state variables are the absolute temperature ϑ, the order parameter χ and its gradient $\nabla\chi$. The thermodynamical laws tell us that the free energy is a concave function with respect to the temperature while there are no constraints concerning the dependence on the other variables. Therefore we choose the functional Ψ of the following form

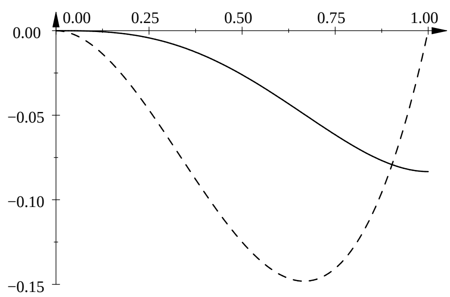
$$\Psi(\vartheta, \chi, \nabla\chi) = -\frac{c_0}{2}\vartheta^2 + F(\chi)\vartheta_c + G(\chi)\vartheta + \frac{\nu}{2} |\nabla\chi|^2 , \tag{2.3}$$

where the constants c_0, ν are positive and $\vartheta_c > 0$ represents the already introduced critical temperature for the phase transition. Indeed we note that the pure caloric part of the free energy $-(c_0/2)\vartheta^2$ is concave.

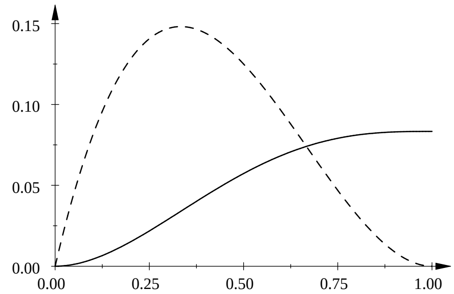
In addition the functions F and G characterise the behaviour of the phase transition. A first order phase transition, for instance in liquid-solid or vapour-liquid system, can be described as follows

$$F(\chi) = \frac{\chi^4}{4} - \frac{\chi^3}{3}, \quad G(\chi) = \frac{\chi^4}{4} - \frac{2\chi^3}{3} + \frac{\chi^2}{2}. \tag{2.4}$$

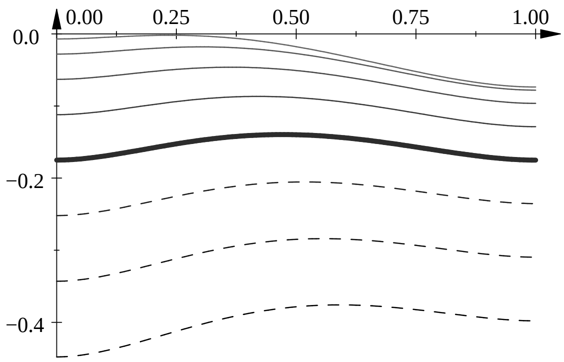
We want to stress out that for low temperatures $\vartheta < \vartheta_c$ the minimum of the free energy Ψ is attained in the pure first phase $\chi = 0$ and for high temperatures $\vartheta > \vartheta_c$ the minimum is attained in the pure second phase $\chi = 1$. The physical admissible range of χ is in this case $[0, 1]$. However in the equilibrium case $\vartheta = \vartheta_c$ there exits two minima in the pure two phases (cf. Figure 2.1).



(a) *F and F' (dashed)*



(b) *G and G' (dashed)*



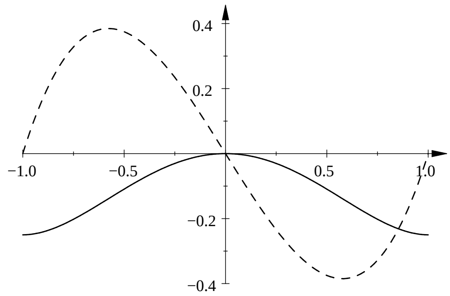
(c) *Free energy Ψ for ϑ from 0.2 up to 1.6 (weak to strong, $\vartheta < \vartheta_c$ solid, $\vartheta = \vartheta_c$ bold, $\vartheta > \vartheta_c$ dashed)*

Figure 2.1.: *Free energy for first order phase transitions ($c_0 = 0.35, \nu = 0.5, \vartheta_c = 1$)*

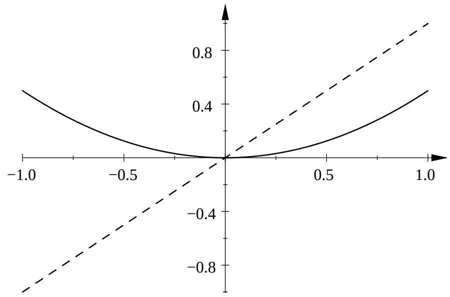
On the other hand for a second order phase transition, for example for superconductivity or ferromagnetism, F and G can be written as

$$F(\chi) = \frac{\chi^4}{4} - \frac{\chi^2}{2}, \quad G(\chi) = \frac{\chi^2}{2}. \tag{2.5}$$

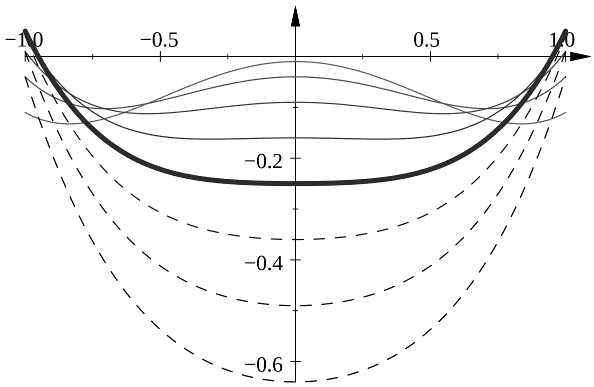
In difference to the first order phase transition the free energy Ψ attains two minima for low temperature $\vartheta < \vartheta_c$ in the mixed phase region once for $\chi \in (-1, 0)$ and once for $\chi \in (0, 1)$. The physical admissible values for χ are here $[-1, 1]$. Although for temperature $\vartheta \geq \vartheta_c$ there is again one minimum attained in $\chi = 0$ (cf. Figure 2.2).



(a) *F and F' (dashed)*



(b) *G and G' (dashed)*



(c) *Free energy Ψ for ϑ from 0.2 up to 1.6 (weak to strong, $\vartheta < \vartheta_c$ solid, $\vartheta = \vartheta_c$ bold, $\vartheta > \vartheta_c$ dashed)*

Figure 2.2.: *Free energy for second order phase transitions ($c_0 = 0.5, \nu = 0.1, \vartheta_c = 1$)*

Remark 2.1 (Admissible values for the phase variable)*.* The physical constraint on χ, that is $\chi_* \leq \chi \leq \chi^*$, is not a priori guaranteed by the choice of the free energy functional Ψ (2.3). Therefore in the literature often an additional term $I_{[\chi_*,\chi^*]}(\chi)$ is used in the definition of Ψ. Hereby $I_{[\chi_*,\chi^*]}$ denotes the subdifferential of the indicator function on the interval $[\chi_*, \chi^*]$, meaning that

$$I_{[\chi_*,\chi^*]}(\chi) = \begin{cases} 0 & \text{, if } \chi \in [\chi_*, \chi^*] \\ +\infty & \text{, if } \chi \notin [\chi_*, \chi^*]. \end{cases} \tag{2.6}$$

The constraint on χ during its evolution is preserved by this modified free energy functional. Our approach did not ask for such a modified energy functional as the constraint on χ is ensured by the evolution of the system itself with certain assumptions on F and G. Hence it will be proved that the equations of the system are somehow *consistent* with respect to the constraint $\chi_* \leq \chi \leq \chi^*$. The proof of these property of our model will be detailed in the sequel and relies on a maximum principle argument.

2.2. Pseudo-potential of dissipation

We introduce a second functional Φ that depends on the dissipative variables χ_t and $\nabla\vartheta$. We choose χ_t, because it is related to microscopic velocities which are responsible for the phase transition, while $\nabla\vartheta$ is concerned with the head flux. In addition the pseudo-potential of dissipation Φ is non-negative, convex with respect to the dissipative variables, and it attains its minimum 0 for a null dissipation, that is when $(\chi_t, \nabla\vartheta) = (0, \mathbf{0})$. Thus we prescribe

$$\Phi(\chi_t, \nabla\vartheta) = \frac{\mu}{2}\,|\chi_t|^2 + \frac{\lambda}{2\vartheta}\,|\nabla\vartheta|^2 \tag{2.7}$$

with μ and λ denoting positive coefficients.

3. System of partial differential equations

3.1. Constitutive relations

Hence, constitutive relations can be written for B, $\mathbf{H}$, s and $\mathbf{Q}$. All these four quantities can be recovered from the free energy, for non dissipative contributions and the pseudo-potential of dissipation, for the dissipative parts. We have

$$B = \frac{\partial\Psi}{\partial\chi} + \frac{\partial\Phi}{\partial\chi_t} = \vartheta_c F'(\chi) + \vartheta G'(\chi) + \mu\chi_t \tag{2.8a}$$

$$\mathbf{H} = \frac{\partial\Psi}{\partial(\nabla\chi)} = \nu\nabla\chi \tag{2.8b}$$

$$s = -\frac{\partial\Psi}{\partial\vartheta} = c_0\vartheta - G(\chi) \tag{2.8c}$$

$$\mathbf{Q} = -\frac{\partial\Phi}{\partial(\nabla\vartheta)} = -\frac{\lambda}{\vartheta}\nabla\vartheta = -\lambda\nabla\log\vartheta. \tag{2.8d}$$

We want to point out that the choice of the free energy (2.3) leads to a linear contribution for the temperature in (2.8c). This preserves sufficient regularity on the solution. In addition that the term $-(c_0/2)\vartheta^2$ could be seen as a first order approximation of the following more general form of the energy potential

$$\Psi(\vartheta, \dots) = -c_0\vartheta\log\vartheta + \dots.$$

In this case, the entropy s would be related to the temperature ϑ through a logarithmic nonlinearity. On the other hand, a logarithmic nonlinearity forcing ϑ to be strictly positive in our expression (2.8d) for the entropy flux $\mathbf{Q}$.

3.2. Initial-boundary value problem

Now, combining the constitute relations (2.8a)-(2.8d) with the evolution equation for the energy (2.1) and the evolutionequation for the entropy (2.2) leads to the following system of partial differential equations with initial and boundary conditions

$$c_0\vartheta_t - G'(\chi)\chi_t - \lambda\Delta\log\vartheta = R(x,t,\vartheta) \tag{2.9a}$$

$$\mu\chi_t - \nu\Delta\chi + F'(\chi)\vartheta_c + G'(\chi)\vartheta = 0 \tag{2.9b}$$

which is addressed in $Q := \Omega \times (0,T)$. For prescribing boundary conditions, we fix Dirichlet condition in $\Gamma \times (0,T)$ for the temperature and a Neumann homogeneous condition for the phase parameter

$$\log\vartheta = \log\vartheta_\Gamma, \quad \partial_n\chi = 0, \tag{2.10}$$

where ∂_n is the external normal derivative. Finally, initial condition are in Ω

$$\vartheta(0) = \vartheta_0, \quad \chi(0) = \chi_0. \tag{2.11}$$

For the sake of simplicity, in the mathematical analysis performed in subsequent sections we will take the physical constants $c_0, \lambda, \mu, \nu, \vartheta_c$ all equal to 1.

3.3. Nonlinearity in the source term

As already mentioned in the entropy balance equation (2.2), we will deal also with a dependence of the source term R on the temperature ϑ. The dependence could be nonlinear and especially singular. We will assume that $R(x,t,\vartheta(x,t))$ is increasing in temperature up to Lipschitz perturbations. Thus we are able to choose the following source terms

$$R(x,t,\vartheta) = \frac{R_1(x,t)}{\vartheta^2} - R_2(x,t)$$

or

$$R(x,t,\vartheta) = R_3(x,t)\vartheta - R_4(x,t),$$

which could be interpreted as a linearization of R around some equilibrium value of ϑ. In such cases, the possible data R_1, R_2 or R_3 and R_4 have to be smooth enough and at least R_1 should be non-negative throughout $\Omega \times (0,T)$.

Chapter 3

Main results

At first in this chapter we introduce carefully the continuous problem, which we later discretise with respect to time. For the time discrete scheme we show the existence and uniqueness results as well as certain regularity and the convergence to the continuous solution. Further, for the continuous problem we reproduce the boundedness result of [3] and extend it with the continuous dependence on the data.

1. Record of the continuous problem

In the sequel Ω is a bounded open set in $\mathbb{R}^3$, whose boundary Γ is assumed to be of class C^2. Moreover, ∂_n is the outward normal derivative on Γ. Given a finite final time T, we set for convenience

$$Q_t := \Omega \times (0,T) \quad \text{for every } t \in (0,T] \quad \text{and} \quad Q := Q_T. \tag{3.1}$$

Next, we describe the structure of our system. Four constants $\vartheta_*, \vartheta^*, \chi_*, \chi^* \in \mathbb{R}$ are given such that

$$0 < \vartheta_* \leq 1 \leq \vartheta^* \quad \text{and} \quad \chi_* < \chi^* \tag{3.2a}$$

and four functions F, G, β and π

$$F, G : \mathbb{R} \to \mathbb{R}, \quad \beta : Q \times (0,\infty) \to \mathbb{R}, \quad \text{and} \quad \pi : Q \times \mathbb{R} \to \mathbb{R}$$

satisfying

$$F, G \in C^2(\mathbb{R}), \quad F \text{ is bounded from below and } G \text{ is nonnegative} \tag{3.2b}$$

$$F', G' < 0 \text{ in } (-\infty, \chi_*), \quad \text{and} \quad F', G' > 0 \text{ in } (\chi^*, \infty) \tag{3.2c}$$

$$\beta \text{ is Lipschitz continuous in } Q \times [\delta, 1/\delta] \text{ for every } \delta \in (0,1) \tag{3.2d}$$

$$\beta_{,x}, \beta_{,t}, \beta' \text{ and } \pi \text{ are Carathéodory functions (see A.1) with the notation} \tag{3.2e}$$

$$\beta_{,x}(x,t,r) := \nabla\beta(x,t,r), \quad \beta_{,t} := \partial_t\beta(x,t,r), \quad \beta'(x,t,r) := \partial_r\beta(x,t,r) \tag{3.2f}$$

$$0 \le \beta'(x,t,r) \le \beta_1(r) \text{ for a.e. } (x,t) \in Q, \forall r \in \mathbb{R} \text{ and some } \beta_1 \in C^0(0,\infty) \tag{3.2g}$$

$$|\beta_{,x}(x,t,r)| + |\beta_{,t}(x,t,r)| \le M_\beta(1 + |\beta(x,t,r)|)$$
$$\qquad \text{for a.a. } (x,t) \in Q, \forall r \in \mathbb{R} \text{ and some } M_\beta \in [0,\infty) \tag{3.2h}$$

$$\beta(x,t,1) = 0 \text{ for every } (x,t) \in Q \tag{3.2i}$$

$$\pi(x,t,r) \text{ is uniformly Lipschitz continuous in } r \text{ for a.a. } (x,t) \in Q \text{ with constant } L_\pi \tag{3.2j}$$

$$\pi(\cdot,\cdot,0) = \pi_0(\cdot,\cdot) \in L^2(Q) \tag{3.2k}$$

Remark 3.1. We note that the bound of β' in (3.2g) and again (3.2i) imply that

$$|\beta(x,t,r)| \le \beta_0(r) := \left| \int_1^r \beta_1(s)\,\mathrm{d}s \right| \quad \text{for all } (x,t,r) \in Q \times (0,\infty). \tag{3.3}$$

Therefore, we recognise by (3.2h) that even $\beta_{,x}$ and $\beta_{,t}$ satisfy an analogous bound and infer that in fact (3.2d) follows from the other assumptions.

Notation 3.2. Let I be a real interval and $\psi : Q \times I \to \mathbb{R}$ be a Carathéodory function according Definition A.1. We use the same symbol ψ also to denote the operator acting on measurable functions on Q as follows. If $v : Q \to I \subset \mathbb{R}$ is measurable

$$\psi(v) \quad \text{denotes the function} \quad (x,t) \mapsto \psi(x,t,v(x,t)), \quad (x,t) \in Q. \tag{3.4a}$$

Note that $\psi(v)$ is measurable due to the Carathéodory assumption on ψ. Similar definitions and symbols are used for functions depending on the space variable. Namely, if $v : \Omega \to I \subset \mathbb{R}$ is measurable

$$\psi(t,v) \quad \text{denotes the function} \quad x \mapsto \psi(x,t,v(x)), \quad x \in \Omega \tag{3.4b}$$

for a.a. $t \in (0,T)$. As well as if $\phi : \Omega \times I \to \mathbb{R}$ is a Carathéodory function

$$\phi(v) \quad \text{denotes the function} \quad x \mapsto \phi(x,v(x)), \quad x \in \Omega. \tag{3.4c}$$

In addition we use the abbreviation

$$H^2_n(\Omega) := \left\{ v \in H^2(\Omega) : \partial_n v = 0 \text{ on } \Gamma \right\}. \tag{3.5}$$

By the way we want to note the important fact that the space $H^{-1}(\Omega)$ is the dual one to $H^1_0(\Omega)$.

Now, we list our assumptions on the boundary and initial data. There are given three functions ϑ_Γ, ϑ_0 and χ_0 be three functions such that

$$\vartheta_\Gamma \in L^2(0,T : H^{1/2}(\Gamma)) \cap H^1(0,T; H^{-1/2}(\Gamma)), \quad \vartheta_* \le \vartheta_\Gamma \le \vartheta^* \quad \text{a.e. on } \Gamma \times (0,T) \tag{3.6a}$$

$$\vartheta_0 \in L^2(\Omega), \quad \vartheta_* \le \vartheta_0 \le \vartheta^* \quad \text{a.e. in } \Omega \tag{3.6b}$$

$$\chi_0 \in H^1(\Omega), \quad \chi_* \le \chi_0 \le \chi^* \quad \text{a.e. in } \Omega \tag{3.6c}$$

where $\vartheta_*, \vartheta^*, \chi_*$ and χ^* are introduced in (3.2a).

Definition 3.3 (Solution of the continuous problem)**.** We say that the triplet (ϑ, χ, ξ) is a solution of the continuous problem if it fulfils

$$\vartheta \in L^\infty(0,T;L^2(\Omega)), \quad \vartheta > 0 \quad \text{a.e. in } Q, \quad \text{and} \quad \ln\vartheta \in L^2(0,T;H^1(\Omega)) \tag{3.7a}$$
$$\chi \in L^2(0,T;H^2_n(\Omega)) \cap H^1(0,T;L^2(\Omega)) \tag{3.7b}$$
$$G(\chi), F'(\chi), G'(\chi) \in L^2(Q) \tag{3.7c}$$
$$\xi \in L^2(Q) \tag{3.7d}$$
$$\partial_t(\vartheta - G(\chi)) \in L^2(0,T;H^{-1}(\Omega)) \tag{3.7e}$$
$$\partial_t(\vartheta - G(\chi)) - \Delta \ln\vartheta + \xi = \pi(\vartheta) \quad \text{in } L^2(0,T;H^{-1}(\Omega)) \quad \text{and} \quad \xi = \beta(\vartheta) \tag{3.7f}$$
$$\partial_t\chi - \Delta\chi + F'(\chi) + G'(\chi)\vartheta = 0 \quad \text{a.e. in } Q \tag{3.7g}$$
$$\ln\vartheta = \ln\vartheta_\Gamma \quad \text{a.e. on } \Gamma \times (0,T) \tag{3.7h}$$
$$(\vartheta - G(\chi))(0) = \vartheta_0 - G(\chi_0) \quad \text{and} \quad \chi(0) = \chi_0 \quad \text{a.e. in } \Omega. \tag{3.7i}$$

Remarks 3.4. (i) Even though ξ is a known function of ϑ, we refer to the triplet (ϑ, χ, ξ) instead of the pair (ϑ, χ), when we speak of a solution.

(ii) Moreover, we note that (3.7a) and (3.7c) yield $G'(\chi)\vartheta \in L^2(0,T;L^1(\Omega))$ and by comparison to (3.7g), even $G'(\chi)\vartheta \in L^2(Q)$.

(iii) We point out that the first condition in (3.7i) reduces to $\vartheta(0) = \vartheta_0$ whenever one knows $G(\chi) \in C^0([0,T];H^{-1}(\Omega))$.

(iv) Actually, some additional smoothness for $G(\chi)$ as well as for $F'(\chi)$ and $G'(\chi)$ surely holds if the nonlinearities satisfy some growth conditions, thanks to (3.7b) or whenever χ is bounded and our existence result stated below ensures such a property.

(v) A homogeneous Neumann boundary condition for χ is entailed by (3.7b) due to the introduced space $H^2_n(\Omega)$.

We quote the existence theorem from [3].

Theorem 3.5 (Existence and maximum principle for χ)**.** *Let the assumptions* (3.2) *be fulfilled and the initial and boundary data satisfy the regularity conditions* (3.6)*. Then, there exists a triplet* (ϑ, χ, ξ) *solving the continuous problem according to the Definition 3.3. Moreover, every solution* (ϑ, χ, ξ) *fulfils the inequalities*

$$\chi_* \le \chi \le \chi^* \quad \textit{a.e. in } Q. \tag{3.8}$$

In particular, χ *is bounded.*

As a consequence of the maximum principle for χ, we can replace F and G by new Lipschitz continuous functions, still termed F and G, satisfying

$$\left|F'\right| \le L_F \quad \text{and} \quad \left|G'\right| \le L_G. \tag{3.9}$$

Indeed we can arbitrarily modify F and G outside $[\chi_*, \chi^*]$.

2. Results for the discrete problem

Before discretising the problem given by the equations (3.7), we introduce some notation.

Definition 3.6 (Partition)**.** A *partition* $\mathcal{P}$ of the interval $[0,T]$ is defined as the ordered set

$$\mathcal{P} := \{t_0 = 0, t_1, \dots, t_{N-1}, t_N = T\}, \text{ where } t_0 < t_1 < \dots < t_N. \tag{3.10}$$

The size of every subinterval is denoted by $\tau_i = t_i - t_{i-1}$ and the *diameter* of the partition is $\tau := \max_i \tau_i$.

The partition $\mathcal{P}$ is said to be *uniform*, if $\min_i \tau_i \geq \sigma\tau$ for a fixed σ with $0 < \sigma \leq 1$.

Notation 3.7. We denote for a set $\left\{\vartheta^i\right\}_{i=1}^N \subset V$ of elements in an arbitrary Banach space V with $\vartheta^{\mathcal{P}}$ the vector

$$\vartheta^{\mathcal{P}} := (\vartheta^1, \dots, \vartheta^N) \in (L^2(\Omega))^N \tag{3.11}$$

Definition 3.8 (Step approximations)**.** Let $\alpha : Q \to \mathbb{R}$ be a locally integrable function and $\mathcal{P}$ denote a partition with diameter τ. Then we define two sets of interpolating points

$$\alpha^i(x) := \alpha(x, t_i) \quad \text{for } i = 0, \dots, N \quad \text{and} \quad \overline{\alpha}^i(x) := \frac{1}{\tau_i}\int_{t_{i-1}}^{t_i} \alpha(x,s)\,\mathrm{d}s \quad \text{for } i = 1, \dots, N. \tag{3.12}$$

In addition we set $\overline{\alpha}^0(x) = \overline{\alpha}^1(x)$.

Remark 3.9. The notation is used for β

$$\beta^i(x,r) = \beta(x, t_i, r)$$

by the above definition. In addition the notation with the bar is used for π and ϑ_Γ. In detail we define for $1 \leq i \leq N$

$$\overline{\pi}^i(\cdot,\cdot) = \frac{1}{\tau^i}\int_{\tau^{i-1}}^{\tau^i} \pi(\cdot, s, \cdot)\,\mathrm{d}s \quad \text{on } \Omega \times \mathbb{R} \text{ and} \quad \overline{\vartheta}_\Gamma^i(\cdot) = \frac{1}{\tau^i}\int_{\tau^{i-1}}^{\tau^i} \vartheta_\Gamma(\cdot, s)\,\mathrm{d}s \quad \text{on } \Gamma.$$

Definition 3.10 (Step and linear interpolations)**.** Take a uniform partition $\mathcal{P}$ with diameter $\tau > 0$. In addition vectors $\vartheta^{\mathcal{P}}$ and $\chi^{\mathcal{P}}$ are given. Then we first denote with

$$\vartheta_\tau(x,t) = \vartheta^i(x) \quad \text{and} \quad \chi_\tau(x,t) = \chi^i(x) \quad \text{for a.a. } (x,t) \in \Omega \times (t_{i-1}, t_i) \tag{3.13a}$$

the *step interpolation* of $\vartheta^{\mathcal{P}}$ and $\chi^{\mathcal{P}}$. In an analogue way we define

$$\begin{aligned} \widehat{\vartheta}_\tau(x,t) &= \frac{t - t_{i-1}}{\tau_i}\vartheta^i(x) + \frac{t_i - t}{\tau^i}\vartheta^{i-1}(x) \qquad \text{for a.a. } (x,t) \in \Omega \times (t_{i-1}, t_i) \\ \widehat{\chi}_\tau(x,t) &= \frac{t - t_{i-1}}{\tau_i}\chi^i(x) + \frac{t_i - t}{\tau^i}\chi^{i-1}(x) \qquad \text{for a.a. } (x,t) \in \Omega \times (t_{i-1}, t_i) \end{aligned} \tag{3.13b}$$

the *linear interpolation* of $\vartheta^{\mathcal{P}}$ and $\chi^{\mathcal{P}}$. As well as for the vector $\beta^{\mathcal{P}} = (\beta^1, \dots, \beta^N)$ we define for a.a. $(x,t,r) \in \Omega \times (t_{i-1}, t_i) \times \mathbb{R}$

$$\beta_\tau(x,t,r) = \beta^i(x,r) \quad \text{and} \quad \widehat{\beta}_\tau(x,t,r) = \frac{t - t_{i-1}}{\tau_i}\beta^i(x,r) + \frac{t_i - t}{\tau^i}\beta^{i-1}(x,r) \tag{3.13c}$$

the *step* and *linear interpolation*. In addition for $\overline{\pi}^{\mathcal{P}} = (\overline{\pi}^1, \dots, \overline{\pi}^N)$ and $\overline{\vartheta}_\Gamma^{\mathcal{P}} = (\overline{\vartheta}_\Gamma^1, \dots, \overline{\vartheta}_\Gamma^N)$ we denote by

$$\begin{aligned} &\overline{\pi}_\tau(x,t,r) = \overline{\pi}^i(x,r) \qquad \text{for a.a. } (x,t,r) \in \Omega \times (t_{i-1}, t_i) \times \mathbb{R} \\ &\overline{\vartheta}_{\Gamma,\tau} = \overline{\vartheta}_\Gamma^i(x) \qquad \text{for a.a. } (x,t) \in \Gamma \times (t_{i-1}, t_i) \end{aligned} \tag{3.13d}$$

the *step interpolations* respecting the mean values.

Remark 3.11. The linear interpolations of ϑ and χ are useful for denoting their backward approximations of the time derivative. Because for $t \in (t_{i-1}, t_i)$ it yields

$$\partial_t \widehat{\vartheta}_\tau = \frac{\vartheta^i - \vartheta^{i-1}}{\tau_i} \quad \text{and} \quad \partial_t \widehat{\chi}_\tau = \frac{\chi^i - \chi^{i-1}}{\tau_i}. \tag{3.14}$$

Now, we are ready for defining the solution of the associated time-discrete problem. Therefore, we replace the time derivatives $\partial_t \vartheta$ and $\partial_t \chi$ in the continuous problem (3.7f) and (3.7g) by their backward approximations as well as the right hand side and boundary data by their step approximations.

Definition 3.12 (Solution of the discrete problem)**.** Let $\mathcal{P}$ denote a uniform partition of the interval $[0,T]$. The triplet $(\vartheta^{\mathcal{P}}, \chi^{\mathcal{P}}, \xi^{\mathcal{P}})$ is called *solution of the discrete problem* if it satisfies for all $1 \le i \le N$

$$\vartheta^i \in L^2(\Omega), \quad \ln \vartheta^i \in H^1(\Omega) \qquad \vartheta^i > 0 \text{ a.e. in } Q \tag{3.15a}$$

$$\chi^i \in H^2_n(\Omega) \cap H^1(\Omega) \quad \text{and} \quad \xi^i \in L^2(\Omega) \tag{3.15b}$$

$$G(\chi^i), F'(\chi^i), G'(\chi^i) \in L^2(\Omega) \tag{3.15c}$$

$$\frac{\vartheta^i - \vartheta^{i-1}}{\tau^i} - G'(\chi^i)\frac{\chi^i - \chi^{i-1}}{\tau^i} - \Delta \ln \vartheta^i + \xi^i = \overline{\pi}^i(\vartheta^{i-1}) \quad \text{and} \quad \xi^i = \beta^i(\vartheta^i) \tag{3.15d}$$

$$\frac{\chi^i - \chi^{i-1}}{\tau^i} - \Delta \chi^i + F'(\chi^i) + G'(\chi^i)\vartheta^{i-1} = 0 \tag{3.15e}$$

and the initial boundary conditions

$$\ln \vartheta^i = \ln \overline{\vartheta}_\Gamma^i \quad \text{a.e. on } \Gamma \tag{3.15f}$$

$$\vartheta^0 = \vartheta_0 \quad \text{and} \quad \chi^0 = \chi_0 \quad \text{a.e. in } \Omega. \tag{3.15g}$$

We can establish an analogue result of Theorem 3.5 for the discrete scheme.

Theorem 3.13 (Existence, uniqueness and boundedness of χ)**.** *Let the assumptions* (3.2) *be fulfilled as well as the initial and boundary data satisfy the regularity conditions* (3.6). *Then there exists a unique triplet* $(\vartheta^{\mathcal{P}}, \chi^{\mathcal{P}}, \xi^{\mathcal{P}})$ *solving the associated discrete problem* (3.15) *according to the Definition 3.12. Moreover, the solution* $(\vartheta^{\mathcal{P}}, \chi^{\mathcal{P}}, \xi^{\mathcal{P}})$ *fulfils the inequalities*

$$\chi_* \le \chi^i \le \chi^* \quad \textit{for all } 1 \le i \le N \textit{ and a.e. in } \Omega. \tag{3.16}$$

In particular, each coordinate of $\chi^{\mathcal{P}}$ *is bounded.*

We will prove this Theorem in Chapter 5.

Theorem 3.14 (Stability result)**.** *There exists a constant c such that for every $\tau > 0$, sufficient small $(\vartheta^{\mathcal{P}}, \chi^{\mathcal{P}}, \xi^{\mathcal{P}})$ is a discrete solution with*

$$\begin{aligned} &\|\vartheta_\tau\|^2_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;W^{1,4/3}(\Omega))} + \left\|\widehat{\vartheta}_\tau\right\|_{H^1(0,T;H^{-1}(\Omega))} + \tau \left\|\widehat{\vartheta}_\tau\right\|^2_{H^1(0,T;L^2(\Omega))} \\ &+ \|\chi_\tau\|^2_{L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega))} + \left\|\widehat{\chi}_\tau\right\|^2_{H^1(0,T;L^2(\Omega))} + \tau \left\|\widehat{\chi}_\tau\right\|^2_{H^1(0,T;H^1(\Omega))} \\ &+ \sup_{t\in[0,T]} \int_\Omega (\vartheta_\tau(t)(\ln\vartheta_\tau(t) - 1) + 1) + \|\ln\vartheta_\tau\|^2_{L^2(0,T;H^1(\Omega))} + \|\beta_\tau(\vartheta_\tau)\|_{L^2(0,T;L^2(\Omega))} \le c. \end{aligned} \tag{3.17}$$

We want to remark that $\xi_\tau = \beta_\tau(\vartheta_\tau)$ in consequence of (3.15d)*.*

We can establish a pointwise convergence based on this theorem.

Theorem 3.15 (Convergence to continuous solution)**.** *Let initial and boundary data be given satisfying* (3.6)*. If we take a sequence of partitions $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ with $\tau_n \to 0$, where τ_n is the diameter of the partition $\mathcal{P}_n$. Then exists a subsequence $\{n_k\}_{k\in\mathbb{N}}$ such that*

$$(\vartheta_{\tau_{n_k}}, \chi_{\tau_{n_k}}, \xi_{\tau_{n_k}}) \to (\vartheta, \chi, \xi) \quad a.e. \ in\ Q \quad as \quad k \to \infty \tag{3.18}$$

and (ϑ, χ, ξ) is a solution of the associated continuous problem.

The proofs of the last two theorems will be shown in Chapter 6.

3. Results for the continuous problem

With the convergence we will reach again at the continuous problem. Here our aim is now to prove a continuous dependence on the data. Therefore, we need in addition to the existence result and the boundedness of χ given in Theorem 3.5 also the boundedness of ϑ. Actually, such a property holds whenever we reinforce the assumption on the structure of our system a little, namely

$$\pi_0 \in L^q(Q) \quad \text{for some } q > 5/2. \tag{3.19}$$

Theorem 3.16 (Boundedness of ϑ)**.** *Assume* (3.19) *in addition to the hypotheses of Theorem 3.5. Then, the component ϑ of any solution (ϑ, χ, ξ) for the continuous problem* (3.7) *is bounded.*

With the boundedness of ϑ we can improve the regularity of ϑ and χ

Proposition 3.17 (Improved regularity)**.** *If the component ϑ of any solution (ϑ, χ, ξ) is bounded, than the following regularities hold*

$$\vartheta^m \in L^2(0,T;H^1(\Omega)) \quad \textit{for every} \quad m \in (0, +\infty). \tag{3.20a}$$

as well as

$$\chi \in L^p(0,T;W^{2,p}(\Omega)) \cap W^{1,p}(0,T;L^p(\Omega)) \quad \textit{for every } p \ge 1. \tag{3.20b}$$

The proof of the Theorem and the Proposition are written down in comparison to [3] in Chapter 7, where a Moser-type procedure is used. Based on this theorem we can finally show the following result.

For the last Theorem we have to reinforce again our assumptions. We set $R(x,t,r) := \pi(x,t,r) - \beta(x,t,r)$ and assume now that R is Lipschitz continuous in the third variable

$$|R(x,t,r_1) - R(x,t,r_2)| \le L_R\,|r_1 - r_2| \qquad \forall r_1, r_2 \in \mathbb{R}, \tag{3.21}$$

where L_R is its Lipschitz constant.

Theorem 3.18 (Continuous dependence on the data)**.** *With the assumptipons of Theorem 3.16 and in addition* (3.21) *we take* $\delta \ge 0$ *and two solutions of the continuous problem* $(\vartheta_i, \chi_i, \xi_i)$ *for* $i = 1, 2$ *with initial and boundary data satisfying*

$$\|\vartheta_{\Gamma,1} - \vartheta_{\Gamma,2}\|_{L^2(0,T;L^2(\Gamma))} \le \delta \tag{3.22a}$$

$$\|\vartheta_{0,1} - \vartheta_{0,2}\|_{H^{-1}(\Omega)} \le \delta \tag{3.22b}$$

$$\|\chi_{0,1} - \chi_{0,2}\|_{H^1(\Omega)} \le \delta. \tag{3.22c}$$

Then exists a constant $c > 0$ *such that*

$$\|\vartheta_1 - \vartheta_2\|_{L^\infty(0,T;H^{-1}(\Omega)) \cap L^2(Q)} + \|\chi_1 - \chi_2\|_{H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))} \le c\delta. \tag{3.23}$$

The proof of this theorem is obtained in Chapter 8.

Chapter 4

Tools

1. Monotone operators

In this section we introduce some notation and deduce some theorems from the theory of monotone operators. They are important in the theory of nonlinear evolution equations. Montone operators can be defined in Banach spaces. This is described by Barbu in [1]. But because we are working in Sobolev spaces, the theory of monotone operators in Hilbert spaces is adequate for our concerns. For this reason we are following up the theory developed by Brézis in [6] and H will be in the sequel a general Hilbert space with a norm denoted by $\|\cdot\|$.

1.1. Definitions and fundamental concepts

Definition 4.1 (Multi-valued operator)**.** A *multi-valued operator* A is a map from $H \to 2^H$, where 2^H is the power set of H. The *domain* of A is denoted by $\mathcal{D}(A) := \{x \in H : Ax \neq \emptyset\}$, as well as the range is called $\mathcal{R}(A) := \bigcup_{x\in H} Ax$. If for every $x \in H$ the set Ax has at most one element, then A is called *single-valued.*

We identify A with his graph in $H \times H$, i.e. $\{(x, y) : y \in Ax\}$. Therewith, the *inverse operator* A^{-1} is the operator with the symmetric graph with respect to A:

$$y \in A^{-1}x \quad \text{if and only if} \quad x \in Ay. \tag{4.1a}$$

Obviously we have $\mathcal{D}(A^{-1}) = \mathcal{R}(A)$.

We can order the set of operators, if we define the *inclusion*:

$$A \subset B \quad \text{if and only if} \quad \forall x \in H : Ax \subset Bx. \tag{4.1b}$$

Definition 4.2 (Monotone operator)**.** An operator A in H is called *monotone*, if for all $x_1, x_2 \in H$ yields

$$\forall y_1 \in Ax_1 \, \forall y_2 \in Ax_2 : (y_1 - y_2, x_1 - x_2) \geq 0. \tag{4.2a}$$

If the operator A is single-valued, we can write more compactly

$$\forall x_1, x_2 \in H : (Ax_1 - Ax_2, x_1 - x_2) \geq 0. \tag{4.2b}$$

We use this formulation also for multi-valued operators and mean then (4.2a).

An operator A in H is said to be *maximal monotone* if the operator A is maximal in the set of monotone operators in relation to $\subset$.

Equivalently, A is a *maximal monotone operator* if and only if A is monotone and for all $(x, y) \in H \times H$ such that

$$\forall \xi \in H : (y - A\xi, x - \xi) \geq 0 \tag{4.2c}$$

follows $y \in Ax$.

Proposition 4.3 (Characterisation of maximal monotonicity)**.** *Let A be an operator in H. The following three properties are equivalent:*

(i) A is maximal monotone.

(ii) A is monotone and $\mathcal{R}(\mathrm{id} + A) = H$.

(iii) For all $\varepsilon > 0$ is $(\mathrm{id} + \varepsilon A)^{-1}$ a contraction defined on whole H.

Proof. See [6, Proposition 2.2., p. 23] □

Definition 4.4 (Hemicontinuity)**.** Let A be a single–valued operator defined on $H \times H$, such that $\mathcal{D}(A) = H$. A is said to be *hemicontinuous* on H if for any $x, y \in H$

$$A(x + ty) \rightharpoonup Ax \quad \text{weakly in } H \quad \text{as} \quad t \to 0. \tag{4.3}$$

Definition 4.5 (Coercivity)**.** The operator $A : H \to 2^H$ is called *coercive* if

$$\lim_{n\to\infty} \frac{(Ax_n, x_n)}{\|x_n\|} = \infty, \tag{4.4}$$

for all sequences $\{x_n\} \subset \mathcal{D}(A)$ such that $\lim_{n\to\infty} \|x_n\| = \infty$.

Proposition 4.6 (Convergence of sequences)**.** *Let A be a maximal monotone operator in H. Take $(x_n, y_n) \in A$ such that*

$$x_n \rightharpoonup x, \quad y_n \rightharpoonup y \quad \textit{in } H \textit{ and } \quad \limsup (y_n, x_n) \leq (y_n, x_n) \quad \textit{as } n \to \infty.$$

Then yields $(x, y) \in A$ and $(y_n, x_n) \to (y, x)$.

Proof. See [6, Proposition 2.5 p. 27]. □

The following corollary will be very useful for showing the existence of solutions to our problem.

Corollary 4.7. *Let B be a monotone and hemicontinuous operator in H and A be a maximal monotone operator in H. Then $A + B$ is maximal monotone. Moreover, if $A + B$ is coercive then $\mathcal{R}(A + B) = H$.*

Proof. See [1, Corollary 1.3 p. 48]. □

1.2. Yosida approximation

The Yosida approximation is an adequate tool for the regularization of maximal monotone operators. We will use the Yosida approximation of the logarithm to handle its singularity. In this way we can obtain an existence result for a regularised form of the equation for ϑ (3.15d).

Definition 4.8 (Yosida approximation)**.** Let $A \subset H \times H$ be a maximal monotone operator. For $\varepsilon > 0$ denotes $J_\varepsilon := (\mathrm{id} + \varepsilon A)^{-1}$ the *resolvent* of A, then

$$A_\varepsilon := \frac{1}{\varepsilon}(\mathrm{id} - J_\varepsilon) \tag{4.5}$$

is called the *Yosida approximation* of A.

Remark 4.9. The Yosida approximation is well-defined as the resolvent J_ε is a contraction in comparison to the Proposition 4.3 (iii).

The following Proposition from [6] gives an overview of the properties of Yosida approximation in general.

Proposition 4.10 (General properties of the Yosida approximation)**.** *Let A be a maximal monotone operator and A_ε denotes its Yosida approximation, then*

(i) A_ε *is a maximal monotone operator and Lipschitz continuous with constant* $\frac{1}{\varepsilon}$.

(ii) $(A_\varepsilon)_\mu = A_{\varepsilon+\mu}$ *for all* $\varepsilon, \mu > 0$.

(iii) *For all* $x \in \mathcal{D}(A)$ *there holds* $\|A_\varepsilon x\| \to \|Ax\|$ *and* $A_\varepsilon x \to Ax$ *when* $\varepsilon \to 0$ *with*

$$\|A_\varepsilon x - Ax\|^2 \leq \|Ax\|^2 - \|A_\varepsilon x\|^2 .$$

(iv) *For* $x \notin \mathcal{D}(A)$, $\|A_\varepsilon x\| \to \infty$ *when* $\varepsilon \to 0$.

Proof. See [6, Proposition 2.6 p. 28]. □

Remark 4.11. From the (iii) of the Proposition follows that $\|A_\varepsilon x\| < \|A\varepsilon\|$ for all $\varepsilon > 0$ in non trivial cases (i.e. A identical to zero). Hence $A_\varepsilon x$ is in fact a regularization with respect to the norm $\|\cdot\|$ in H.

Finally, we use a result stating in which case the limit of a converging sequence of Yosida approximated elements is an element of the regularised maximal monotone operator.

Proposition 4.12. *Let H denote a Hilbert space, $A : H \to 2^H$ a maximal monotone operator and A_ε its Yosida approximation. In addition take $\{\varepsilon_n\}$ a positive sequence converging to zero and $\{x_n\} \subset H$ such that exist $x, y \in H$ with*

$$x_n \rightharpoonup x \quad \text{and} \quad A_{\varepsilon_n} x_n \rightharpoonup y \quad \text{in } H, \quad \limsup_{n\to\infty}(A_{\varepsilon_n} x_n, x_n) \leq (y, x).$$

Then $y \in Ax$.

Proof. We define $y_n := A_{\varepsilon_n} x_n$ and $x_n' := J_{\varepsilon_n} x_n$, where J_ε is the resolvent of A. Then yields according the Definition of the inverse (4.1a)

$$x_n' + \varepsilon_n A x_n' \ni x_n.$$

But with the Definition 4.8 of A_ε follows

$$y_n = \frac{x_n - x_n'}{\varepsilon_n} \quad \text{as} \quad x_n' = x_n - \varepsilon_n y_n$$

and with the help of the hypothesis we obtain $x_n' \rightharpoonup x$ in H. Now follows

$$\limsup_{n\to\infty} (y_n, x_n') = \limsup_{n\to\infty} (y_n, x_n - \varepsilon_n y_n) \leq (y, x)$$

and we can conclude using Proposition 4.6. □

Definition 4.13 (Regularization of ln)**.** In our case the resolvent $\rho_\varepsilon : \mathbb{R} \to \mathbb{R}_+$ of the logarithm is given by the transcendental equation

$$\rho_\varepsilon(r) + \varepsilon \ln \rho_\varepsilon(r) = r. \tag{4.6a}$$

Corresponding to Definition 4.8 the *Yosida approximation of the logarithm* is

$$\ln_\varepsilon r := \frac{r - \rho_\varepsilon(r)}{\varepsilon}. \tag{4.6b}$$

Later we use the altered function $\mathrm{Ln}_\varepsilon : \mathbb{R} \to \mathbb{R}$

$$\mathrm{Ln}_\varepsilon r := \varepsilon r + \ln_\varepsilon r \tag{4.6c}$$

and its primitive

$$\mathcal{L}_\varepsilon(r) := \int_1^r \mathrm{Ln}_\varepsilon s \, \mathrm{d}s = \frac{\varepsilon}{2}(r^2 - 1) + \int_1^r \ln_\varepsilon s \, \mathrm{d}s. \tag{4.6d}$$

Proposition 4.14 (Properties of the resolvent ρ_ε)**.** *The resolvent defined in* (4.6a) *allows the representation*

$$\rho_\varepsilon(r) = \exp\left(\frac{r}{\varepsilon} - \mathrm{W}\left(\frac{1}{\varepsilon} e^{\frac{r}{\varepsilon}}\right)\right), \tag{4.7a}$$

where W*, called* Lambert W function*, fulfils the equation* $z = \mathrm{W}(z)e^{\mathrm{W}(z)}$*. In the representation we use the first real branch of* W*.*

Additionally, ρ_ε *is for every fixed* $\varepsilon > 0$ *a strictly monotone, convex and positive function. Especially the first derivative admits the estimate*

$$\rho_\varepsilon'(r) \leq 1 - \frac{\varepsilon}{\max(1, r) + \varepsilon}. \tag{4.7b}$$

Proof. We substitute $\rho_\varepsilon = e^{\phi_\varepsilon}$ in (4.6a) and get

$$e^{\phi_\varepsilon} + \varepsilon \phi_\varepsilon = r.$$

Now, setting $t = \frac{r}{\varepsilon} - \phi_\varepsilon$ and some transformations result in

$$te^t = -\frac{1}{\varepsilon}e^{\frac{r}{\varepsilon}}.$$

By using the Definition of Lambert's W function and substituting t back, we obtain

$$\phi_\varepsilon(r) = \frac{r}{\varepsilon} - \mathrm{W}\left(\frac{1}{\varepsilon}e^{\frac{r}{\varepsilon}}\right),$$

which gives the representation by definition of ϕ_ε.

The positivity of ρ_ε for fixed ε directly follows from the representation (4.7a), because we choose the real branch of W. We calculate the first derivative implicitly. Therefore, we differentiate the equation (4.6a)

$$\rho'_\varepsilon(r) + \varepsilon\frac{\rho'_\varepsilon(r)}{\rho_\varepsilon(r)} = 1$$

and thereby

$$\rho'_\varepsilon(r) = \frac{\rho_\varepsilon(r)}{\rho_\varepsilon + \varepsilon} = 1 - \frac{\varepsilon}{\rho_\varepsilon(r) + \varepsilon} > 0 \tag{4.8}$$

as ρ_ε is positive. We remark that obviously also $\rho'_\varepsilon(r) \leq 1$. Further differentiating leads to

$$\rho''_\varepsilon(r) = \frac{\rho'_\varepsilon(r)}{\rho_\varepsilon(r) + \varepsilon}\left(1 - \frac{\rho^2_\varepsilon(r)}{(\rho_\varepsilon(r) + \varepsilon)^2}\right) \geq 0$$

by the same argument. Thus ρ_ε is also convex.

Obviously, we have $\rho_\varepsilon(1) = 1$ for all $\varepsilon > 0$. Thereby, with the monotonicity and the bound of the derivative follows

$$\rho_\varepsilon(r) \leq \max(1, r),$$

which completes the proof by equation (4.8). □

Proposition 4.15 (Properties of the approximations $\ln_\varepsilon$ and Ln_ε)**.** *We have*

$$\ln_\varepsilon^{-1}(s) = e^s + \varepsilon s \quad \textit{for every } s \in \mathbb{R} \tag{4.9a}$$

$$\frac{\ln r}{1+\varepsilon} \leq \ln_\varepsilon r \leq \ln r \quad \textit{for every } r \geq 1 \tag{4.9b}$$

$$\mathrm{Ln}'_\varepsilon(r) \geq 1 \quad \textit{for every } r \leq 1 \quad \textit{and} \quad \mathrm{Ln}'_\varepsilon(r) \geq \frac{1}{2r} \quad \textit{for every } r > 1 \tag{4.9c}$$

We set $l_* := \min\{0, \ln\vartheta_*\}$ *and* $l^* := \max\{0, \ln\vartheta^*\}$*, then we have*

$$l_* \leq \ln_\varepsilon r \leq l^* \quad \textit{for every } r \in [\vartheta_*, \vartheta^*] \tag{4.9d}$$

Moreover, we have

$$\frac{1}{2\vartheta^*} \leq \ln'_\varepsilon(r) \leq \frac{2}{\vartheta_*} \quad \textit{for every } r \in [\vartheta_*, \vartheta^*] \tag{4.9e}$$

for ε *small enough.*

(4.9f)

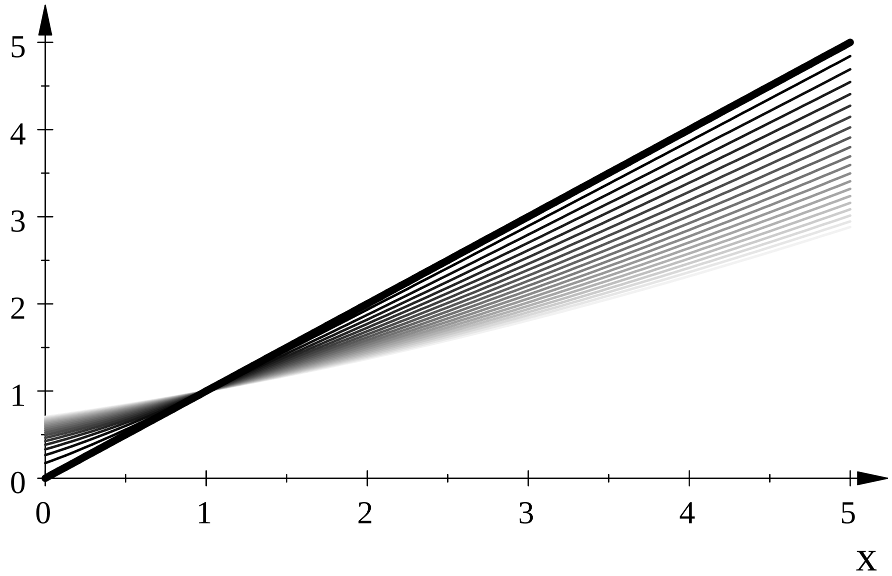


Figure 4.1.: *Resolvent of the Yosida approximation of the logarithm* $\varepsilon = 0.1, \dots, 2$ *(weak to strong) and the identity (bold)*

Proof. Take any $s \in \mathbb{R}$. Then $r = e^s + \varepsilon s$ satisfies $\rho_\varepsilon(r) = e^s$ by (4.6a). Now, (4.9a) follows by the Definition of the Yosida approximation 4.8

$$\ln_\varepsilon r = \frac{r - \rho_\varepsilon(r)}{\varepsilon} = \frac{e^s + \varepsilon s - e^s}{\varepsilon} = s. \tag{4.10a}$$

At first we observe for $s \geq 0$

$$e^{(1+\varepsilon)s} \geq e^s + \varepsilon s \geq e^s \tag{4.10b}$$

If $r \geq 1$, applying $s := \ln_\varepsilon r$ to (4.9a) leads to $r = e^{\ln_\varepsilon r} + \varepsilon \ln_\varepsilon r$. On the other hand using the inequality (4.10b) we obtain

$$e^{(1+\varepsilon)\ln_\varepsilon r} \geq r \geq e^{\ln_\varepsilon r}.$$

And (4.9b) follows from applying the logarithm.

Firstly, we need

$$\ln_\varepsilon'(r) = \frac{1 - \rho_\varepsilon'(r)}{\varepsilon} = \frac{1}{\varepsilon}\left(1 - \frac{\rho_\varepsilon(r)}{\rho_\varepsilon(r) + \varepsilon}\right) = \frac{1}{\rho_\varepsilon(r) + \varepsilon}. \tag{4.10c}$$

Assuming $\rho_\varepsilon(r) > 1$ in (4.6a) implies $r > \rho_\varepsilon(r) > 1$ and hence $\rho_\varepsilon(r) \leq 1$ for $r \leq 1$. We conclude that

$$\mathrm{Ln}_\varepsilon'(r) = \varepsilon + \ln_\varepsilon'(r) \geq \varepsilon + \frac{1}{1+\varepsilon} = \frac{1 + \varepsilon + \varepsilon^2}{1 + \varepsilon} \geq 1$$

for every $r \leq 1$. If we now assume $r > 1$, then $1 < \rho_\varepsilon < r$. Taking (4.10c) into account, we infer that

$$\mathrm{Ln}_\varepsilon'(r) \geq \varepsilon + \frac{1}{r + \varepsilon} \geq \frac{1}{2r} \quad \text{if } \varepsilon \leq r.$$

Let us take the observations $e^{l_*} + \varepsilon l_* \leq \vartheta_*$ and $e^{l^*} + \varepsilon l^* \geq \vartheta^*$ into account. They lead to $e^{l_*} + \varepsilon l_* \leq r \leq e^{l^*} + \varepsilon l^*$ for every $r \in [\vartheta_*, \vartheta^*]$. By using (4.9a) property 4.9d is obtained.

With the strictly monotonicity of ρ_ε and (4.10c) follows that for $r \in [\vartheta_*, \vartheta^*]$

$$\frac{1}{\rho_\varepsilon(\vartheta^*) + \varepsilon} \leq \ln'_\varepsilon(r) \leq \frac{1}{\rho_\varepsilon(\vartheta_*) + \varepsilon}.$$

In addition, it is clear that $\rho_\varepsilon(r') \to r'$ as $\varepsilon \to 0$ for every $r' > 0$. Then (4.9e) follows immediately if ε is small enough. □

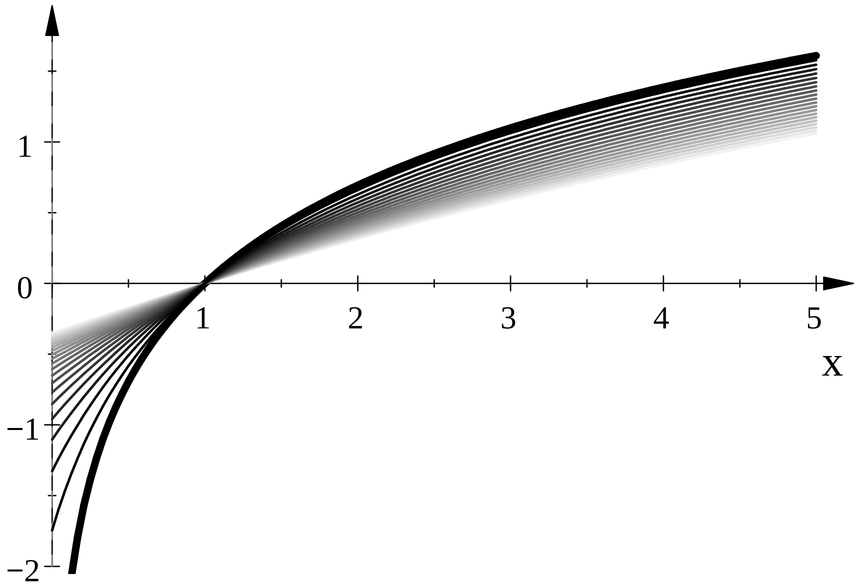


Figure 4.2.: *Yosida approximation of the logarithm for* $\varepsilon = 0.1, \dots, 2$ *(weak to strong) and the logarithm (bold)*

Remark 4.16. $\mathcal{L}_\varepsilon(r)$ is a convex function, which is bounded from below uniformly with respect to ε, since $\ln_\varepsilon 1 = 0$.

2. Subdifferential mappings

In this section we introduce notation and prove some lemma, which are important to understand the proof of the boundedness of ϑ in Chapter 7. Nevertheless we are still working in Hilbert spaces, we want to mark the more general results obtained in [1] for Banach spaces, where all not proven results could be found. Therefore, X denote in the sequel a real Banach space with its dual X^*. In addition we denote by $\langle \cdot, \cdot \rangle$ the dual pairing in $X \times X^*$ or $X^* \times X$ depending on the case more comfortable. We mark the elements belonging to X^* also with an asterisk * to distinguish them from the elements of X.

Definition 4.17 (Proper convex, lower-semicontinuous and conjugate functions)**.** A *proper convex function* on X is a function $\phi : X \to (-\infty, +\infty]$, not identically $+\infty$, such that

$$\phi((1-\lambda)x + \lambda y) \leq (1-\lambda)\phi(x) + \lambda\phi(y) \quad \text{for all } x, y \in X \text{ and } 0 \leq \lambda \leq 1. \tag{4.11a}$$

The function $\phi : X \to (-\infty, +\infty]$ is said to be *lower-semicontinuous* on X if

$$\liminf_{y \to x} \phi(y) \geq \phi(x) \quad \text{for any } x \in X. \tag{4.11b}$$

In addition the function ϕ^* on X^* defined by

$$\phi^*(x^*) := \sup_{x \in X} \{\langle x, x^* \rangle - \phi(x)\} \tag{4.11c}$$

is called the *(convex) conjugated* of ϕ.

The convex conjugated is well defined in the following sense and also it is a conjugation on the class of lower-semicontinuous proper convex functions.

Proposition 4.18. *Let ϕ be a lower-semicontinuous proper convex function on X. Then the conjugate ϕ^* is convex, lower-semicontinuous and proper on the dual space X^*. In addition there holds $\phi^{**} = \phi$.*

Definition 4.19 (Subgradient and subdifferential)**.** Given a proper convex function ϕ on X and a point $x \in X$, then we denote by $\partial\phi(x)$ the set of all $x^* \in X^*$ such that

$$\phi(x) \leq \phi(y) + \langle x - y, x^* \rangle, \quad \text{for every } y \in X. \tag{4.12}$$

Such elements x^* are called *subgradients* of ϕ at x and $\partial\phi(x)$ is called the *subdifferential* of ϕ at x.

Remark 4.20. We observe immediately from the definition that $\partial\phi$ is a monotone operator from X to X^*. More precise $\partial\phi$ is a maximal monotone operator if X is a real Banach space. This result was obtained by Rockafellar in [23]. In addition $\partial\phi$ is always a closed convex set. In a similar way like for the gradient, the condition $0 \in \partial\phi(x)$ is necessary and sufficient for $\phi(x) = \min_{y \in X} \phi(y)$.

Proposition 4.21. *If Φ on X is a lower-semicontinuous proper convex function, then it holds*

$$\partial(\Phi^*) = (\partial\Phi)^{-1} \tag{4.13}$$

$$
\begin{array}{ccc}
\Phi & \xrightarrow{\partial} & \partial\Phi \\
{\scriptstyle *}\big\downarrow & & \big\downarrow{\scriptstyle -1} \\
\Phi^* & \xrightarrow{\partial} & \partial(\Phi^*) = (\partial\Phi)^{-1}
\end{array}
$$

Figure 4.3.: *Commutative diagram of the Proposition 4.21*

Proof. Fix x^* and take $x \in \partial\Phi^*(x^*)$. Then x satisfies according to (4.12)

$$\Phi^*(x^*) \leq \Phi^*(y^*) + \langle x, x^* - y^* \rangle \quad \text{for all } y^* \in X^*$$

Now, by using the Definition 4.17 of the conjugate function results in

$$\langle z, x^* \rangle - \Phi(z) \leq \sup_{\overline{z} \in X} \{\langle \overline{z}, y^* \rangle - \Phi(\overline{z})\} + \langle x, x^* - y^* \rangle \quad \text{for all } z \in X \text{ and all } y^* \in X^*.$$

Rearranging the terms a little bit holds

$$\inf_{\overline{z} \in X} \{\Phi(\overline{z}) - \langle \overline{z} - x, y^* \rangle\} \leq \Phi(z) + \langle x - z, x^* \rangle \quad \text{for all } z \in X \text{ and all } y^* \in X^*.$$

The infimum exists, as Φ is proper convex. In addition the infimum has to hold for every $y^* \in X^*$, thus we can always choose $y^* = -c(\overline{z} - x)^*$ with $c > 0$. Then by contradiction the infimum is obtained for $\overline{z} = x$ due to the lower-semicontinuity of Φ. Finally, we have obtained

$$\Phi(x) \leq \Phi(z) + \langle x - z, x^* \rangle \quad \text{for all } z \in X,$$

hence $x^* \in \partial\Phi(x)$ and therefore $x \in (\partial\Phi)^{-1}(x^*)$. □

We need two more specific results to prove the main Lemma of this section. The first Proposition gives a construction under which conditions subdifferentials of integrals are induced by their integrands.

Definition 4.22 (Convex integrands)**.** Let $g : \mathbb{R} \to (-\infty, +\infty]$ be a lower-semicontinuous proper convex function on $\mathbb{R}$ and let $\beta = \partial g$. Let $\phi : L^2(\Omega) \to (-\infty, +\infty]$ be defined by

$$\phi(u) = \begin{cases} \int_\Omega g(u(x))\,\mathrm{d}x & \text{if } g(u) \in L^1(\Omega) \\ +\infty & \text{otherwise} \end{cases} \tag{4.14}$$

where Ω is a bounded domain in $\mathbb{R}^n$.

Proposition 4.23. *The function ϕ is lower-semicontinuous and convex on $L^2(\Omega)$. Moreover, $w \in \partial\phi(u)$ if and only if $w(x) \in \beta(u(x))$ a.e. in Ω and*

$$\overline{\mathcal{D}(\phi)} = \left\{ u \in L^2(\Omega) : u(x) \in \overline{\mathcal{D}(g)} \ \textit{a.e. on } \Omega \right\} \tag{4.15}$$

The next Lemma points out which regularity of a function u and the subdifferential $\partial\phi(u)$ of a lower-semicontinuous proper convex function ϕ is necessary to obtain absolute continuity of the concatenation of $\phi(u)$.

Lemma 4.24. *Let $u \in H^1(0,T;H^{-1}(\Omega))$ such that $u(t) \in \mathcal{D}(\partial\phi)$ a.e. in $(0,T)$. If there exists $g \in L^2(0,T;H_0^1(\Omega))$ such that $g(t) \in \partial\phi(u(t))$ a.e. in $(0,T)$, then the function $t \mapsto \phi(u(t))$ is absolutely continuous in $[0,T]$.*

Further there holds

$$\partial_t \phi(u(t)) = \langle \partial_t u(t), v \rangle \quad \forall v \in \partial\phi(u(t)) \text{ a.e. in } (0,T). \tag{4.16}$$

Proof. Similar to the proof in [6, Lemme 3.3, p. 73]. □

Now, we can state the result: a weak general chain rule. Therewith, we can handle the lack of regularity for a time derivative $\partial_t\vartheta$, which is known only to belong to $L^2(0,T;H^{-1}(\Omega))$.

Lemma 4.25 (Weak chain rule for time derivatives)**.** *Assume*

$$\vartheta \in L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)) \quad \text{and} \quad \vartheta > 0 \quad \text{a.e. in } Q.$$

Moreover, let $\phi : (0,\infty) \to \mathbb{R}$ be a convex function of class C^1 and assume that

$$\phi'(\vartheta) \in L^2(0,T;H_0^1(\Omega)).$$

If $\Phi : \mathbb{R} \to (-\infty,\infty]$ denotes the extension

$$\Phi(r) := \phi(r) \quad \text{if } r > 0, \quad \Phi(0) := \lim_{r\to 0^+} \phi(r), \quad \text{and} \quad \Phi(r) := +\infty \quad \text{if } r < 0, \tag{4.17}$$

then the function $t \mapsto \int_\Omega \Phi(\vartheta(t))$ is absolutely continuous on $[0,T]$ and we have

$$\int_0^t \big\langle \partial_t\vartheta(s), \phi'(\vartheta(s)) \big\rangle \,\mathrm{d}s = \int_\Omega \Phi(\vartheta(t)) - \int_\Omega \Phi(\vartheta(0)) \quad \text{for every } t \in [0,T] \tag{4.18}$$

Proof. We first observe the Φ is the convex, proper and lower-semicontinuous extension of ϕ on $\mathbb{R}$. In addition, we notice that

$$\phi'(u) \in \partial\Phi(u) \quad \text{a.e. in } \Omega \quad \text{if} \quad u \in L^2(\Omega), \quad u > 0 \text{ a.e. in } \Omega, \quad \text{and} \quad \phi'(u) \in L^2(\Omega). \tag{4.19a}$$

With the help of the Proposition 4.21 we know that Φ and its conjugated Φ^* satisfy $\partial\Phi^* = (\partial\Phi)^{-1}$. For a.e. $t \in (0,T)$ we set for convenience $v(t) := \phi'(\vartheta(t))$ and observe that both $\vartheta(t)$ and $v(t)$ are in $L^2(\Omega)$ as well as $\vartheta(t) > 0$ by the assumptions. Hence, we just achieved that $v(t) \in \partial\Phi(\vartheta(t))$ a.e. in Ω by (4.19a) and consequently by Proposition 4.21

$$\vartheta(t) \in \partial\Phi^*(v(t)) \quad \text{a.e. in } \Omega. \tag{4.19b}$$

Moreover, defining the functionals $J : H \to (-\infty,+\infty]$ and $J_0 : H_0^1(\Omega) \to (-\infty,+\infty]$ according to the Definition 4.22 as follows

$$J(v) := \begin{cases} \int_\Omega \Phi^*(v) & \text{if } \Phi^*(v) \in L^1(\Omega) \\ +\infty & \text{otherwise} \end{cases} \qquad J_0 := J|_{H_0^1(\Omega)}. \tag{4.19c}$$

From the Proposition 4.23 follows that J is convex, proper and lower-semicontinuous. In addition its subdifferential $\partial J \subset L^2(\Omega) \times L^2(\Omega)$ is exactly induced by $\partial\Phi^*$ via the almost everywhere inclusion in Ω. Then, for a.a. $t \in (0,T)$ by (4.19b) follows $\vartheta \in \partial J(v(t))$ meaning

$$J(v(t)) \leq J(w) + \langle \vartheta(t), v(t) - w \rangle \quad \forall w \in L^2(\Omega).$$

Therefore, as $v(t) \in H_0^1(\Omega)$ we have that

$$J_0(v(t)) \leq J_0(w) + (\vartheta(t), v(t) - w) \quad \forall w \in L^2(\Omega),$$

whence the functional $J_0 : H_0^1(\Omega) \to (-\infty, +\infty]$ is proper and the inclusion $\vartheta(t) \in \partial J_0(v(t))$ holds for the subdifferential $\partial J_0 \subset H^{-1}(\Omega) \times H_0^1(\Omega)$. Introducing the conjugate functionals and the subdifferentials

$$\begin{aligned} J^* : L^2(\Omega) \to (-\infty, +\infty], \quad & J_0^* : H^{-1}(\Omega) \to (-\infty, +\infty] \\ \text{and} \quad \partial J^* \subset L^2(\Omega) \times L^2(\Omega), \quad & \partial J_0^* \subset H^{-1}(\Omega) \times H_0^1(\Omega), \end{aligned}$$

and recalling from the Proposition 4.21 that $\partial J^* = (\partial J)^{-1}$ as well as $\partial J_0^* = (\partial J_0)^{-1}$, we observe that $v(t) \in \partial J^*(\vartheta(t))$ and $v(t) \in \partial J_0^*(\vartheta(t))$ for a.a. $t \in (0,T)$. We conclude by applying the previous Lemma 4.24 that $J_0^*(\vartheta)$ is absolutely continuous and

$$\int_0^t \big\langle \partial_t \vartheta(s), \phi'(\vartheta(s)) \big\rangle \, \mathrm{d}s = \int_0^t \langle \partial_t \vartheta(s), v(s) \rangle \, \mathrm{d}s = J_0^*(\vartheta(t)) - J_0^*(\vartheta(0)) \quad \forall t \in [0,T]. \tag{4.19d}$$

From Proposition 4.18 follows that for $u, w \in L^2(\Omega)$ one has $w \in \partial J^*(u)$ if and only if $w \in \partial\Phi(u)$ a.e. in Ω. If we can proof in addition

$$J_0^*(u) = J^*(u) = \int_\Omega \Phi(u) \quad \text{if } u \in L^2(\Omega) \text{ and } J_0^*(u) \leq +\infty. \tag{4.19e}$$

Then, since ϑ is weakly continuous from $[0,T]$ to $L^2(\Omega)$, we can conclude from (4.19d), the absolute continuity of $J_0^*(\vartheta)$ and (4.19e) that the assertion is true.

It remains to check (4.19e). For $u \in L^2(\Omega)$ holds

$$J_0^*(u) = \sup_{w \in H_0^1(\Omega)} \{\langle u, w \rangle - J_0(w)\} \leq \sup_{w \in L^2(\Omega)} \{\langle u, w \rangle - J(w)\} = J^*(u).$$

On the other hand, in view of [9, Lemma 2.3] it turns out that for all $w \in L^2(\Omega)$ there exists a sequence $\{w_n\} \subset H_0^1(\Omega)$ such that $w_n \to w$ in $L^2(\Omega)$ and $J_0(w_n) = J(w_n) \to J(w)$, whence

$$\langle u, w \rangle - J(w) = \lim_{n \to \infty} (\langle u, w_n \rangle - J(w_n)) = \lim_{n \to \infty} (\langle u, w_n \rangle - J_0(w_n)) \leq J_0^*(u) \tag{4.19f}$$

and consequently $J^*(u) \leq J_0^*(u)$ as (4.19f) holds for all $w \in L^2(\Omega)$. Then, (4.19e) turns out to be true and the proof is completely finished. □

3. Time approximation

Definition 4.26 (Translation operator)**.** We define for $k \in \mathbb{Z}$ the *translation operator* $\mathcal{T}_k$ acting on approximations α_τ on a partition $\mathcal{P}$. Take $t \in (t_{i-1}, t_i)$, then $\mathcal{T}_k$ is defined by

$$\mathcal{T}_k[\alpha_\tau](t) := \begin{cases} \alpha(0) & \text{for } k+i \leq 0 \\ \alpha\left(\left(\frac{t-t_{i-1}}{t_i}\right)\tau_{i-k} + t_{i-k-1}\right) & \text{for } 0 < k+i < N \\ \alpha(t_n) & \text{for } k+i \geq N. \end{cases} \tag{4.20}$$

Lemma 4.27. *Let $\mathcal{P}$ denote a uniform partition with diameter τ. In addition $\varphi : \mathbb{R} \to [0, \infty)$ is a convex function and $\alpha : Q \to \mathbb{R}$ is locally integrable, then*

$$(\varphi \circ \overline{\alpha}_\tau)(x,t) \leq \overline{(\varphi \circ \alpha)}_\tau(x,t) \quad \textit{for all } (x,t) \in Q, \tag{4.21}$$

in which $\circ$ denotes the composition of two functions $f \circ g := f(g)$.

Proof. Fix $x \in \Omega$ and take $t \in (t_{i-1}, t_i)$ for some $1 \leq i \leq N$. Then

$$\begin{aligned} \varphi(\overline{\alpha}^i(x)) &= \varphi\left(\frac{1}{\tau_i}\int_{t_{i-1}}^{t_i} \alpha(x,s)\,\mathrm{d}s\right) = \varphi\left(\int_0^1 \alpha(x, \tau_i\widetilde{s} + t_{i-1})\,\mathrm{d}\widetilde{s}\right) \\ &\leq \int_0^1 \varphi\left(\alpha(x, \tau_i\widetilde{s} + t_{i-1})\right)\,\mathrm{d}\widetilde{s} = \frac{1}{\tau_i}\int_{t_{i-1}}^{t_i} \varphi\left(\alpha(x,s)\right)\,\mathrm{d}s = \overline{\varphi(\alpha)}^i(x,t), \end{aligned}$$

where the estimate is a consequence of the Jensen inequality A.2. □

Proposition 4.28. *Take $v, w : Q \to \mathbb{R}$ with $v \in L^2(0,T;V)$ and $w \in H^1(0,T;V)$, where V is an arbitrary Hilbert space over Ω. We denote by $\overline{v}_\tau$ the step approximation of v and by $\widehat{\overline{w}}_\tau$ respectively $\widehat{w}_\tau$ the linear approximations with respect to the mean respectively end points of w on a uniform partition $\mathcal{P}$. Then*

$$\|\overline{v}_\tau\|_{L^2(0,T;V)} \leq \|v\|_{L^2(0,T;V)} \tag{4.22a}$$

as well as

$$\|\partial_t \widehat{w}_\tau\|_{L^2(0,T;V)} \leq \|\partial_t w\|_{L^2(0,T;V)} \quad \textit{and} \quad \left\|\widehat{\overline{w}}_\tau\right\|_{H^1(0,T;V)} \leq \left(1 + \frac{1}{\sigma}\right)\|w\|_{H^1(0,T;V)}, \tag{4.22b}$$

where σ is the uniformity constant of the partition $\mathcal{P}$ introduced in the Definition 3.6

Proof. For the first one we conclude

$$\|\overline{v}_\tau\|^2_{L^2(0,T;V)} = \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \|\overline{v}^i\|_V^2 \overset{(4.21)}{\leq} \sum_{i=1}^N \tau_i \frac{1}{\tau_i}\int_{t_{i-1}}^{t_i} \|v(\cdot,s)\|_V^2\,\mathrm{d}s \leq \|v\|^2_{L^2(0,T;V)}.$$

On the other hand we have

$$\begin{aligned} \|\partial_t \widehat{w}_\tau\|^2_{L^2(0,T;V)} &= \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \left\|\frac{w(\cdot,t_i) - w(\cdot,t_{i-1})}{\tau_i}\right\|_V^2 = \sum_{i=1}^N \tau_i \left\|\frac{1}{\tau_i}\int_{t_{i-1}}^{t_i} \partial_t w(\cdot,s)\,\mathrm{d}s\right\|_V^2 \\ &\overset{(4.21)}{\leq} \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \|\partial_t w\|_V^2 = \|\partial_t w\|^2_{L^2(0,T;V)}. \end{aligned}$$

The last estimate we split into the terms $\left\|\widehat{\overline{w}}_\tau\right\|^2_{L^2(0,T;V)}$ and $\left\|\partial_t\widehat{\overline{w}}_\tau\right\|^2_{L^2(0,T;V)}$. For the first one it yields

$$\begin{aligned}
\left\|\widehat{\overline{w}}_\tau\right\|^2_{L^2(0,T;V)} &= \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \left\| \frac{t-t_{i-1}}{\tau_i}\overline{w}^i + \frac{t_i - t}{\tau_i}\overline{w}^{i-1}\right\|_V^2 \\
&\leq \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \left[2\left(\frac{t-t_{i-1}}{\tau_i}\right)^2 \left\|\overline{w}^i\right\|_V^2 + 2\left(\frac{t_i-t}{\tau_i}\right)^2 \left\|\overline{w}^{i-1}\right\|_V^2 \right] \\
&\leq \sum_{i=1}^N \frac{2}{3}\tau_i \left(\left\|\overline{w}^i\right\|_V^2 + \left\|\overline{w}^{i-1}\right\|_V^2\right) \\
&\leq \sum_{i=2}^N \left(\frac{\tau_i}{\tau_i}\int_{t_{i-1}}^{t_i} \left\|w(\cdot,s)\right\|_V^2 \,\mathrm{d}s + \frac{\tau_i}{\tau_{i-1}}\int_{t_{i-2}}^{t_{i-1}} \left\|w(\cdot,s)\right\|_V^2 \,\mathrm{d}s\right) + \int_{t_0}^{t_1} \left\|w(\cdot,s)\right\|_V^2 \,\mathrm{d}s \\
&\leq \left(1+\frac{1}{\sigma}\right)\left\|w\right\|^2_{L^2(0,T;V)},
\end{aligned}$$

where we used the uniformity of the partition $\mathcal{P}$, since

$$\frac{\tau_i}{\tau_{i-1}} \leq \frac{\tau}{\sigma\tau} = \frac{1}{\sigma}.$$

Finally we conclude for the other seminorm using the Mean Value Theorem of Integration

$$\begin{aligned}
\left\|\partial_t\widehat{\overline{w}}_\tau\right\|^2_{L^2(0,T;V)} &= \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \left\| \frac{\overline{w}^i - \overline{w}^{i-1}}{\tau^i}\right\|_V^2 \\
&= \sum_{i=2}^N \frac{1}{\tau_i}\left\| \frac{1}{\tau_i}\int_{t_{i-1}}^{t_i} w(s)\,\mathrm{d}s - \frac{1}{t_{i-1}}\int_{t_{i-2}}^{t_{i-1}} w(s)\,\mathrm{d}s\right\|_V^2 \\
s_i \in (t_{i-1}, t_i) \text{ for } i = 1,..,N \qquad &= \sum_{i=2}^N \frac{1}{\tau^i}\left\|\frac{w(s_i)-w(s_{i-1})}{\tau_i}\right\|_V^2 = \sum_{i=2}^N \frac{1}{\tau_i}\left\|\int_{s_{i-1}}^{s_i} \partial_t w(s)\,\mathrm{d}s\right\|_V^2 \\
&\leq \sum_{i=2}^N \frac{s_i - s_{i-1}}{\tau_i}\int_{s_{i-1}}^{s_i} \left\|\partial_t w(s)\right\|_V^2 \,\mathrm{d}s \leq \left(1+\frac{1}{\sigma}\right)\left\|\partial_t w\right\|^2_{L^2(0,T;V)},
\end{aligned}$$

where we used in the last step the following estimate

$$\frac{s_i - s_{i-1}}{\tau_i} \leq \frac{t_i - t_{i-2}}{t_i - t_{i-1}} = \frac{t_i - t_{i-1} + t_{i-1} - t_{i-2}}{t_i - t_{i-1}} = 1 + \frac{\tau_{i-1}}{\tau_i} \leq 1 + \frac{1}{\sigma}.$$

□

Lemma 4.29. *Take $z \in L^2(Q)$ and let $\pi : Q \times \mathbb{R} \to \mathbb{R}$ fulfil the assumptions* (3.2e) *and* (3.2j)*. Furthermore, we denote by $\overline{\pi}_{\tau^n}$ and $\overline{z}_{\tau^n}$ their step approximations on a family of uniform partitions $\{\mathcal{P}_n\}$ such that $\tau^n \to 0$ as $n \to \infty$. Then there holds*

$$\overline{\pi}_{\tau^n}(\overline{z}_{\tau^n}) \to \pi(z) \quad \text{in } L^2(Q) \quad \text{as } n \to \infty. \tag{4.23}$$

Proof. Firstly, we observe that

$$\|\overline{z}_{\tau^n}(\cdot,t) - z(\cdot,t)\|_{L^2(\Omega)} \to 0 \quad \text{for a.a. } t \in (0,T) \quad \text{as} \quad n \to \infty,$$

due to the Lebesgue-Besicovitch Differentiation Theorem A.4. Hence there also holds

$$\|\overline{z}_{\tau^n}(\cdot,t) - z(\cdot,t)\|_{L^2(Q)} \to 0 \quad \text{as} \quad n \to \infty,$$

with the help of the Proposition 4.28 and the Lebesgue dominated convergence Theorem. Now we use the triangle inequality

$$\|\overline{\pi}_{\tau^n}(\overline{z}_{\tau^n}) - \pi(z)\|^2_{L^2(Q)} \leq \|\overline{\pi}_{\tau^n}(\overline{z}_{\tau^n}) - \pi(\overline{z}_{\tau^n})\|^2_{L^2(Q)} + \|\pi(\overline{z}_{\tau^n}) - \pi(z)\|^2_{L^2(Q)}. \tag{4.24}$$

For the first norm we can apply again the Lebesgue-Besicovitch Differentiation Theorem A.4 and hence

$$\frac{1}{\tau^n_{i_n}} \int_{t^n_{i_n-1}}^{t^n_{i_n}} \left|\pi(x,s,z^{i_n}(x)) - \pi(x,t,z^{i_n}(x))\right|^2 \mathrm{d}s \to 0 \quad \text{for a.a. } (x,t) \in Q \quad \text{as} \quad n \to \infty,$$

where the sequence $\{i_n\}$ is chosen depending on t such that $t \in (t_{i_n-1}, t_{i_n})$. Thus the convergence holds also in $L^2(Q)$.

The second norm in (4.24) can be estimated with the help of the uniform Lipschitz continuity assumption on π

$$\|\pi(\overline{z}_{\tau^n}) - \pi(z)\|^2_{L^2(Q)} \leq L_\pi^2 \, \|\overline{z}_{\tau^n} - z\|_{L^2(Q)}.$$

Finally, we have achieved the convergence

$$\|\overline{\pi}_{\tau^n}(\overline{z}_{\tau^n}) - \pi(z)\|^2_{L^2(Q)} \to 0 \quad \text{as} \quad n \to \infty.$$

□

Lemma 4.30. *Take $z \in L^2(Q)$ and let $\beta : Q \times (0,\infty) \to \mathbb{R}$ fulfil the almost uniform Lipschitz continuity assumption* (3.2d). *In addition β_{τ^n} and $\overline{z}_{\tau^n}$ denote their step approximations in the endpoints respectively mean points on a family of uniform partitions $\{\mathcal{P}_n\}$ such that $\tau^n \to 0$ as $n \to \infty$. Then there holds*

$$\beta_{\tau^n}(\overline{z}_{\tau^n}) \to \beta(z) \quad \textit{a.e. in } Q \textit{ and weakly in } L^2(Q) \quad \textit{as} \quad n \to \infty. \tag{4.25}$$

Proof. To gain the almost everywhere convergence, it suffices to show that for every $\delta \in (0,1)$ we have

$$\beta_{\tau^n}(\overline{z}_{\tau^n}) \to \beta(z) \quad \text{almost uniformly in } Q^\delta := \{(x,t) \in Q : \delta \leq z(x,t) \leq 1/\delta\},$$

whenever $\overline{z}_\tau \to z$ a.e. in Q. Thus fix $\delta \in (0,1)$ and take $\eta > 0$. Then we have to show that a subset $Q^\delta_\eta \subset Q^\delta$ exists such that $\left|Q^\delta \backslash Q^\delta_\eta\right| \leq \eta$ and $\beta_{\tau^n}(\overline{z}_{\tau^n}) \to \beta(z)$ uniformly in Q^δ_η, where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^4$. With the help of the Severini-Egorov Theorem we find $Q^\delta_\eta \subset Q^\delta$ such that $\left|Q^\delta \backslash Q^\delta_\eta\right| \leq \eta$ and $\overline{z}_{\tau^n} \to z$ uniformly in Q^δ_η. But therewith we can also prove that $\beta_{\tau^n}(\overline{z}_{\tau^n}) \to \beta(z)$ uniformly in Q^δ_η.

We remark that β is uniform Lipschitz continuous on Q^δ_η with $\operatorname{lip}\beta =: L_{\beta,\delta}$. Than we observe

$$\|\beta_{\tau^n}(\overline{z}_{\tau^n}) - \beta(\overline{z}_{\tau^n})\|_{L^2(Q^\delta_\eta)} \leq \|\beta_{\tau^n}(\overline{z}_{\tau^n}) - \beta(\overline{z}_{\tau^n})\|_{L^2(Q^\delta_\eta)} \leq L_{\beta,\delta}\tau^n$$

Therewith yields

$$\begin{aligned}\|\beta_{\tau^n}(\overline{z}_{\tau^n}) - \beta(z)\|^2_{L^2(Q^\delta_\eta)} &\leq \|\beta_{\tau^n}(\overline{z}_{\tau^n}) - \beta(\overline{z}_{\tau^n})\|^2_{L^2(Q^\delta_\eta)} + \|\beta(\overline{z}_{\tau^n}) - \beta(z)\|^2_{L^2(Q^\delta_\eta)} \\ &\leq L^2_{\beta,\delta}(\tau^n)^2 + L^2_{\beta,\delta}\,\|\overline{z}_{\tau^n} - z\|^2_{L^2(Q)}\,.\end{aligned}$$

and deduce that $\beta_{\tau^n}(\overline{z}_{\tau^n})$ converges to $\beta(z)$ uniformly on Q^δ_η. Hence we have obtained the almost everywhere convergence, which is implied by the almost uniform convergence. Due to the last norm estimate and the just proved almoster everywhere convergence we can apply Lemma A.5, which conclude the proof. □

4. Approximation by extension, regularization and truncation

In this section another procedure for regularising functions is described. We extend the domain of the function by a periodic extension. Afterwards we can apply a convolution with a C^∞-function and restrict the domain again.

Let $\beta : Q \times (0,\infty) \to \mathbb{R}$ fulfil the assumptions (3.2d)-(3.2g). Then take a partition $\mathcal{P}$ with diameter τ and define on that the linear approximation $\widehat{\beta}_\tau$ for the endpoints according the Definition 3.8.

4.1. Extension

Definition 4.31 (Extension operator)**.** We define the both domains:

$$\Omega_\varepsilon := \{x \in \Omega : \operatorname{dist}(x,\Gamma) < \varepsilon\} \quad \text{and} \quad \Omega'_\varepsilon := \left\{x \in \mathbb{R}^3 \backslash \overline{\Omega} : \operatorname{dist}(x,\Gamma) < \varepsilon\right\}.$$

Then there exists $\varepsilon_0 \in (0,1)$ such that for every $x \in \Omega'_{\varepsilon_0}$ we can find a unique point $\widetilde{x} \in \Omega_{\varepsilon_0}$ satisfying

$$x' := \frac{x + \widetilde{x}}{2} \in \Gamma \quad \text{and } x - \widetilde{x} \text{ is orthogonal to } \Gamma \text{ at } x'. \tag{4.26a}$$

The correspondence $x \mapsto \widetilde{x}$ is a bi-Lipschitz diffeomorphism of class C^1 (as Γ is of class C^2) from Ω'_{ε_0} onto Ω_{ε_0}.

Now, we set $\widetilde{\Omega} := \overline{\Omega} \cup \Omega'_{\varepsilon_0}$. We define with $v \in L^\infty(\Omega)$ the extension operator

$$\mathcal{E} : L^\infty(\Omega) \to L^\infty(\widetilde{\Omega}), \qquad \mathcal{E}v(x) := \begin{cases} v(x) & \text{if } x \in \Omega \\ v(\widetilde{x}) & \text{if } x \in \Omega'_{\varepsilon_0} \end{cases} \tag{4.26b}$$

Proposition 4.32 (Properties of $\mathcal{E}$)**.** *The extension operator $\mathcal{E}$ is linear and continuous. Moreover one has*

$$\sup_{\widetilde{\Omega}} \operatorname{ess} \mathcal{E}v = \sup_{\Omega} \operatorname{ess} v \quad \textit{and} \quad \inf_{\widetilde{\Omega}} \operatorname{ess} \mathcal{E}v = \inf_{\Omega} \operatorname{ess} v \quad \textit{for every } v \in L^\infty(\Omega). \tag{4.27a}$$

And also the following properties apply to all $v \in L^\infty(\Omega)$:

$$\mathcal{E} \geq 0 \text{ a.e. in } \widetilde{\Omega} \text{ whenever } v \geq 0 \text{ a.e. in } \Omega \tag{4.27b}$$

$$\|\nabla \mathcal{E}v\|_{L^\infty(\widetilde{\Omega})} \leq M \|\nabla v\|_{L^\infty(\Omega)} \text{ if } \nabla v \in L^\infty(\Omega) \tag{4.27c}$$

$$\operatorname{lip}(\mathcal{E}v) \leq M \operatorname{lip} v \text{ if } v \text{ is Lipschitz continuous,} \tag{4.27d}$$

for some constant M, where $\operatorname{lip} v$ *is the Lipschitz constant of v.*

Proof. Linearity is obvious. For the continuity with $v \in L^\infty(\Omega)$ we have

$$\|\mathcal{E}v\|_{L^\infty(\widetilde{\Omega})} = \|v\|_{L^\infty(\Omega)}$$

and hence $\|\mathcal{E}\|_{L^\infty(\Omega)\to L^\infty(\widetilde{\Omega})} = 1$. Therewith, the estimates (4.27a) and (4.27b) are just consequences. The last two properties are consequences of the bi-Lipschitz continuity of the diffeomorphism $\widetilde{\cdot}$ and M is its Lipschitz constant. □

At this point, we define for $\varepsilon \in (0,1)$ the functions $\widetilde{\beta}^i, \widetilde{\beta}^i_\varepsilon : \widetilde{\Omega} \times \mathbb{R} \to \mathbb{R}$ by using the extension operator $\mathcal{E}$

$$\widetilde{\beta}^i(x,r) := \left(\mathcal{E}\beta^i(\cdot,r)\right)(x) \tag{4.28a}$$

$$\widetilde{\beta}^i_\varepsilon(x,r) := \widetilde{\beta}^i(x,r_\varepsilon) \quad \text{for } i = 1,\dots,N, \text{ where } r_\varepsilon := \max\{\varepsilon, \min\{r, 1/\varepsilon\}\}. \tag{4.28b}$$

In all of the following statements on $\widetilde{\beta}^i$, $\widetilde{\beta}^i_\varepsilon$ and later β^i_ε we do not stress out explicitly every time for all $i = 1,\dots,N$.

We observe that $\widetilde{\beta}^i_\varepsilon$ is globally Lipschitz continuous. Indeed, by recalling (3.2d) as well as (4.27d) and setting for convenience

$$L_\delta := \operatorname{lip} \beta|_{Q\times[\delta,1/\delta]} \quad \text{for } \delta \in (0,1) \tag{4.29}$$

we conclude $\operatorname{lip} \widetilde{\beta}^i_\varepsilon \le M L_\varepsilon$ by the Definition (4.28b). Moreover, as $\mathcal{E}$ is linear and thanks to (4.27b) we infer that both $\widetilde{\beta}(x,t,\cdot)$ and $\widetilde{\beta}_\varepsilon(x,t,\cdot)$ are nondecreasing on $\mathbb{R}$ for every $x \in \widetilde{\Omega}$. Furthermore, both $\widetilde{\beta}^i$ and $\widetilde{\beta}^i_\varepsilon$ vanish at $r = 1$ due to (3.2i). In particular, their values at every $r \in \mathbb{R}$ have the sign of $r - 1$.

4.2. Regularization and truncation

Finally, we are ready to define a C^∞–approximation $\beta^i_\varepsilon : \Omega\times\mathbb{R} \to \mathbb{R}$ of β^i. We regularise $\widetilde{\beta}^i_\varepsilon$ by convolution and restrict the regularization we obtain to $\Omega \times \mathbb{R}$. Namely, we fix a nonnegative $\zeta \in C^\infty(\mathbb{R}^4)$ supported in the unit ball U of $\mathbb{R}^4$ and normalised in $L^1(\mathbb{R}^4)$. Then, by assuming $\varepsilon_0 \le T$ and $\varepsilon \in (0,\varepsilon_0)$ (such restrictions are not stressed in the sequel, but it is understood that they are satisfied), we recall (4.29) and set

$$\delta_\varepsilon := \frac{\varepsilon}{1+L_\varepsilon} \quad \text{and} \quad \zeta_\varepsilon(x,r) := \delta_\varepsilon^{-4}\zeta\left((x,r)/\delta_\varepsilon\right) \quad \text{for } (x,r) \in \mathbb{R}^4 \tag{4.30a}$$

$$\beta^i_\varepsilon(x,r) := (\widetilde{\beta}^i_\varepsilon * \zeta_\varepsilon)(x,r) = \int_{\delta_\varepsilon U} \widetilde{\beta}^i_\varepsilon(x-y, r-s)\,\zeta_\varepsilon(y,s)\,\mathrm{d}y\,\mathrm{d}s$$

$$= \int_U \widetilde{\beta}^i_\varepsilon(x-\delta_\varepsilon y, r-\delta_\varepsilon s)\,\zeta(y,s)\,\mathrm{d}y\,\mathrm{d}s \quad \text{for } (x,r) \in \Omega\times\mathbb{R}. \tag{4.30b}$$

4.3. Properties

The reason for the above choice of δ_ε is that we would like to have

$$|\beta^i_\varepsilon(x,r) - \widetilde{\beta}^i_\varepsilon(x,r)| \le M\varepsilon \quad \text{for every } (x,r) \in \Omega\times\mathbb{R} \tag{4.31}$$

and some constant M. Actually, (4.31) holds with the constant M that makes (4.27d) true, as we show at once. Indeed we have

$$|\beta^i_\varepsilon(x,r) - \widetilde{\beta}^i_\varepsilon(x,r)| = \left|\int_U \left(\widetilde{\beta}^i_\varepsilon(x-\delta_\varepsilon y, r-\delta_\varepsilon s) - \widetilde{\beta}^i_\varepsilon(x,r)\right)\,\zeta(y,s)\,\mathrm{d}y\,\mathrm{d}s\right|$$

$$\le \int_U ML_\varepsilon\,|(\delta_\varepsilon y, \delta_\varepsilon s)|\,\zeta(y,s)\,\mathrm{d}y\,\mathrm{d}s \le ML_\varepsilon\delta_\varepsilon \le M\varepsilon$$

since ML_ε is a Lipschitz constant for $\widetilde{\beta}^i_\varepsilon$, as just observed. With a similar argument, we see that

$$\beta^i_\varepsilon \text{ is Lipschitz continuous with} \quad \operatorname{lip} \beta^i_\varepsilon \le ML_\varepsilon \tag{4.32}$$

since such a property holds for $\widetilde{\beta}^i_\varepsilon$. In the sequel we use the following more precise facts

$$\sup_{\Omega\times[\delta,1/\delta]} |\beta^i_\varepsilon| \le \sup_{\widetilde{\Omega}\times[\delta/2,1/\delta+\delta/2]} |\widetilde{\beta}^i| \le \sup_{\Omega\times[\delta/2,1/\delta+\delta/2]} |\beta(\cdot,t_i,\cdot)| \tag{4.33a}$$

$$\operatorname{lip} \beta^i_\varepsilon|_{\Omega\times[\delta,1/\delta]} \le \operatorname{lip} \widetilde{\beta}^i|_{\widetilde{\Omega}\times[\delta/2,1/\delta+\delta/2]} \le \operatorname{lip} \beta|_{\Omega\times(t_{i-1},t_i]\times[\delta/2,1/\delta+\delta/2]}, \tag{4.33b}$$

for $\delta \in (0,1)$ and $\varepsilon \le \delta/2$. Indeed, we have $\delta_\varepsilon \le \varepsilon \le \delta/2$. Hence, if $(x,t) \in Q$ and $\delta \le r \le 1/\delta$, the values of $\widetilde{\beta}^i_\varepsilon$ in (4.30b) actually are values of $\widetilde{\beta}^i$ at points of the set $\widetilde{\Omega} \times [\delta/2, 1/\delta + \delta/2]$, where $\widetilde{\beta}^i$ is bounded and Lipschitz continuous. Therefore, both the supremum (4.33a) and the Lipschitz constant (4.33b) are preserved by the convolution since ζ is normalised. Finally, we point out that

$$\beta^i_\varepsilon(x,\cdot) \text{ is nondecreasing on } \mathbb{R} \text{ for every } x \in \Omega \tag{4.34}$$

since such a property holds for $\widetilde{\beta}^i_\varepsilon$ and ζ is nonnegative. Moreover, we set for convenience

$$B^i_\varepsilon(x,r) := \int_1^r \beta_\varepsilon(x,s)\,\mathrm{d}s \quad \text{for a.a. } x \in \Omega \text{ and every } r \in \mathbb{R}. \tag{4.35a}$$

Clearly, B^i_ε is convex with respect to the second variable. Furthermore, as $\widetilde{\beta}^i_\varepsilon(x,r)$ and $r-1$ have the same sign as just observed, we see that (4.31) implies

$$B^i_\varepsilon(x,r) \ge \int_1^r \left(\beta^i_\varepsilon(x,s) - \widetilde{\beta}^i_\varepsilon(x,s)\right)\mathrm{d}s \ge -M\varepsilon|r-1| \quad \text{for every } (x,r) \in \Omega\times\mathbb{R}. \tag{4.35b}$$

Finally, it is clear that the first of (4.33a) implies the analogue for B^i_ε, namely

$$\sup_{\Omega\times[\delta,1/\delta]} |B^i_\varepsilon| \le c_\delta \quad \text{for every } \delta \in (0,1) \tag{4.35c}$$

and some constant c_δ.

Proposition 4.33. *We have*

$$|\beta^i_{\varepsilon,x}(x,r)| \le c(1 + |\beta^i_\varepsilon(x,r)|) \tag{4.36}$$

for every $(x,r) \in \Omega\times\mathbb{R}$*, some constant* c *and* ε *small enough.*

Proof. We first prove that

$$|\widetilde{\beta}^i_{\varepsilon,x}(x,r)| \le MM_\beta(1 + |\widetilde{\beta}^i_\varepsilon(x,r)|) \tag{4.37}$$

for every $x \in \widetilde{\Omega}$ and $r \in \mathbb{R}$, where M and M_β are constants satisfying (3.2h) respectively (4.27c) and (4.27d). Firstly we assume $r \in [\varepsilon, 1/\varepsilon]$. Then, $\widetilde{\beta}^i_\varepsilon = \widetilde{\beta}^i$ due to (4.28b) and (4.37) coincides with the analogue for $\widetilde{\beta}^i$. But this easily follows from the inequality for β^i (3.2h) and the property of the extension operator (4.27c) as well as the definition of $\widetilde{\beta}^i$ (4.28a). In the case $r < \varepsilon$ we can argument

$$|\nabla\widetilde{\beta}^i_\varepsilon(x,r)| = |\widetilde{\beta}^i_{,x}(x,t,\varepsilon)| \le MM_\beta(1 + |\widetilde{\beta}^i(x,\varepsilon)|) \le MM_\beta(1 + |\widetilde{\beta}^i_\varepsilon(x,r|).$$

A similar argument holds for $r > 1/\varepsilon$ and (4.37) is established.

In order to prove now (4.33), we notice that

$$\pm\beta^i_{\varepsilon,x} = \pm\nabla(\widetilde{\beta}^i_\varepsilon * \zeta_\varepsilon) = \pm(\widetilde{\beta}^i_{\varepsilon,x}) * \zeta_\varepsilon \leq c(1 + |\widetilde{\beta}^i_\varepsilon| * \zeta_\varepsilon) = c + c|\widetilde{\beta}^i_\varepsilon| * \zeta_\varepsilon$$

with $c := MM_\beta$, since (4.37) holds and the convolution with the nonnegative normalised kernel ζ_ε preservers order and constants. Therefore, we have to bound the last convolution with the right-hand side of the inequality (4.33), which we want to prove. Assume at first $x \in \Omega$ and $r \geq 1 + \varepsilon$. Then, $\widetilde{\beta}^i_\varepsilon(y, s) \geq 0$ for $y \in \widetilde{\Omega}$ and $|s - r| \leq \delta_\varepsilon$ (since $\delta_\varepsilon \leq \varepsilon$, see (4.30a)) and we have

$$(|\widetilde{\beta}^i_\varepsilon| * \zeta_\varepsilon)(x, r) = (\widetilde{\beta}^i_\varepsilon * \zeta_\varepsilon)(x, r) = \beta^i_\varepsilon(x, r) = |\beta^i_\varepsilon(x, r)|.$$

If $r < 1 - \varepsilon$ the argument is similar. Finally, if $|r - 1| \leq \varepsilon$, by assuming $\varepsilon \leq 1/4$, we have

$$\delta_\varepsilon \leq 1/4, \quad r - \delta_\varepsilon > \max\{\varepsilon, 1/2\} \quad \text{and} \quad r + \delta_\varepsilon \leq \min\{1/\varepsilon, 3/2\},$$

whence

$$\left|\left(|\widetilde{\beta}^i_\varepsilon| * \zeta_\varepsilon\right)(x, r)\right| = \left|\left(|\widetilde{\beta}^i| * \zeta_\varepsilon\right)(x, r)\right| \leq \sup_{\widetilde{\Omega}\times[r-\delta_\varepsilon, r+\delta_\varepsilon]} |\widetilde{\beta}^i| \leq \sup_{\widetilde{\Omega}\times[1/2,3/2]} |\widetilde{\beta}^i|$$

since ζ_ε is normalised in $L^1(\mathbb{R}^4)$ and so the hypothesis follows. □

Lemma 4.34. *Assume $z, z_n \in L^2(\Omega), z > 0$ a.e. in Ω, and $z_n \to z$ a.e. in Ω. Moreover, let $\{\varepsilon_n\}$ be a positive real sequence converging to 0. Then, $\{\beta^i_{\varepsilon_n}(z_n)\}$ converges to $\beta^i(z)$ a.e. in Ω.*

Proof. It suffices to show that for every $\delta \in (0, 1)$ we have

$$\beta^i_{\varepsilon_n}(z_n) \to \beta^i(z) \quad \text{almost uniformly in } \Omega^\delta := \{x \in \Omega : \delta \leq z(x) \leq 1/\delta\}.$$

For applying the Severini-Egorov Theorem we fix $\delta \in (0, 1)$ and $\eta > 0$. Therewith, we find $\Omega^\delta_\eta \subset \Omega^\delta$ such that $\left|\Omega\backslash\Omega^\delta_\eta\right| \leq \eta$ and $z_n \to z$ uniformly in Ω^δ_η. Now fix $\overline{n}$ such that

$$\varepsilon_n \leq \frac{\delta}{2} \qquad \frac{\delta}{2} \leq z_n \leq \frac{2}{\delta} \quad \text{in } \Omega^\delta_\eta \text{ for every } n \geq \overline{n}.$$

On the other hand, we have

$$\left\|\beta^i_{\varepsilon_n}(z_n) - \beta^i(z)\right\|_{L^\infty(\Omega^\delta_\eta)} \leq \left\|\beta^i_{\varepsilon_n}(z_n) - \beta^i_{\varepsilon_n}(z)\right\|_{L^\infty(\Omega^\delta_\eta)} + \left\|\beta^i_{\varepsilon_n}(z) - \beta^i(z)\right\|_{L^\infty(\Omega^\delta_\eta)}.$$

Assume now $n \geq \overline{n}$. Then, $\varepsilon_n \leq \delta$, whence $\varepsilon_n \leq z \leq 1/\varepsilon_n$. Thus, $\beta^i(z) = \widetilde{\beta}^i_{\varepsilon_n}(z)$ by the truncation procedure (4.28). We infer that the last term is bounded by $M\varepsilon_n$ as a consequence of (4.32). On the other hand, as $\varepsilon_n \leq \delta/2$. Further as $\varepsilon_n \leq \delta/2$ we can use the bound of the Lipschitz constants in (4.33b). Finally, we conclude that

$$\left\|\beta^i_{\varepsilon_n}(z_n) - \beta^i(z)\right\|_{L^\infty(\Omega^\delta_\eta)} \leq c_\delta \left\|z_n - z\right\|_{L^\infty(\Omega^\delta_\eta)} + M\varepsilon_n$$

and deduce that $\beta^i_{\varepsilon_n}(z_n)$ converges to $\beta^i(z)$ uniformly in Ω^δ_η. □

5. Harmonic extension

The last utility handles the relationships between the approximating nonlinearities and the boundary datum ϑ_Γ. For applying the Poincaré inequality in the next sections we need a known smooth function that coincides with the step approximation of ϑ_Γ on the boundary. Thus, a natural choice is the harmonic extension of $\overline{\vartheta}_{\Gamma,\tau}$.

Definition 4.35 (Harmonic extension of $\overline{\vartheta}_{\Gamma,\tau}$)**.** Let $\mathcal{P}$ be a uniform partition with diameter τ. Then the vector $\vartheta^{\mathcal{P}} = (\vartheta^1_{\mathcal{H}}, \dots, \vartheta^N_{\mathcal{H}})$ is defined by

$$\vartheta^i_{\mathcal{H}} \in H^1(\Omega), \quad \vartheta^i_{\mathcal{H}}|_\Gamma = \overline{\vartheta}^i_\Gamma, \quad \Delta\vartheta^i_{\mathcal{H}} = 0 \quad \text{in } \Omega, \quad \text{for } i = 1 \dots N. \tag{4.38}$$

The step $\vartheta_{\mathcal{H},\tau}$ respectively linear $\widehat{\vartheta}_{\mathcal{H},\tau}$ interpolation of the vector $\vartheta^{\mathcal{P}}$ are the harmonic extensions in the time approximated boundary values $\overline{\vartheta}_{\Gamma,\tau}$ respectively $\widehat{\overline{\vartheta}}_{\Gamma,\tau}$.

Proposition 4.36. *Let ϑ_Γ fulfil the assumption* (3.6a)*. Then one has $\vartheta_{\mathcal{H},\tau} \in L^2(0,T;H^1(\Omega))$ with the estimate*

$$\begin{aligned} \|\vartheta_{\mathcal{H},\tau}\|_{L^2(0,T;H^1(\Omega))} &\le c\,\|\vartheta_\Gamma\|_{L^2(0,T:H^{1/2}(\Gamma))} \\ \text{and} \quad \vartheta_* \le \vartheta_{\mathcal{H},\tau} &\le \vartheta^* \quad \text{a.e. in } Q. \end{aligned} \tag{4.39a}$$

as well as $\widehat{\vartheta}_{\mathcal{H},\tau} \in H^1(0,T;L^2(\Omega))$ bounded by

$$\begin{aligned} \left\|\widehat{\vartheta}_{\mathcal{H},\tau}\right\|_{H^1(0,T;L^2(\Omega))} &\le c\,\|\vartheta_\Gamma\|_{H^1(0,T;H^{-1/2}(\Gamma))} \\ \text{and} \quad \vartheta_* \le \widehat{\vartheta}_{\mathcal{H},\tau} &\le \vartheta^* \quad \text{a.e. in } Q. \end{aligned} \tag{4.39b}$$

Proof. Due to the maximum principle for elliptic equations we find that $\vartheta_* \le \vartheta^i_{\mathcal{H}} \le \vartheta^*$ a.e. in Ω for all $i = 1, \dots, N$. Thus the same bounds are valid for $\vartheta_{\mathcal{H},\tau}$ and $\widehat{\vartheta}_{\mathcal{H},\tau}$.

Further it holds $\|\vartheta^i_{\mathcal{H}}\|_{H^1(\Omega)} \le \left\|\overline{\vartheta}^i_\Gamma\right\|_{H^{1/2}(\Omega)}$, due to the general theory of elliptic equations.

Now let us prove that $\vartheta_{\mathcal{H},\tau} \in L^2(0,T;H^1(\Omega))$ with the given bound.

$$\|\vartheta_{\mathcal{H},\tau}\|^2_{L^2(0,T;H^1(\Omega))} = \sum_{i=1} \int_{t_{i-1}}^{t_i} \|\vartheta^i_{\mathcal{H}}\|^2_{H^1(\Omega)} \le \|\overline{\vartheta}_{\Gamma,\tau}\|^2_{L^2(0,T;H^{1/2}(\Gamma))} \le \|\vartheta_\Gamma\|^2_{L^2(0,T;H^{1/2}(\Gamma))},$$

where the last estimate is a consequence of (4.22a).

In addition we conclude $\|\vartheta^i_{\mathcal{H}}\|_{L^2(\Omega)} \le \left\|\overline{\vartheta}^i_\Gamma\right\|_{H^{-1/2}(\Omega)}$ from the theory of elliptic equations. Due to the linearity of the Laplace equation (4.38) we even have

$$\left\|\frac{t-t_{i-1}}{\tau_i}\vartheta^i_{\mathcal{H}} + \frac{t_i - t}{\tau_i}\vartheta^{i-1}_{\mathcal{H}}\right\|^2_{H^{-1/2}(\Gamma)} \le \left\|\frac{t-t_{i-1}}{\tau_i}\overline{\vartheta}^i_\Gamma + \frac{t_i - t}{\tau_i}\vartheta^{i-1}_\Gamma\right\|^2_{H^{-1/2}(\Gamma)} \qquad \forall i = 1, \dots, N \text{ and } t \in (t_{i-1}, t_i).$$

Thereby we can estimate the $L^2(0,T;L^2(\Omega))$-Norm of $\widehat{\vartheta}_{\mathcal{H},\tau}$.

$$\begin{aligned}\left\|\widehat{\vartheta}_{\mathcal{H},\tau}\right\|^2_{L^2(0,T;L^2(\Omega))} &= \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \left\| \frac{t-t_{i-1}}{\tau_i}\vartheta^i_{\mathcal{H}} + \frac{t_i - t}{\tau_i}\vartheta^i_{\mathcal{H}} \right\|^2_{L^2(\Omega)} \\ &\leq \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \left\| \frac{t-t_{i-1}}{\tau_i}\overline{\vartheta}^i_{\Gamma} + \frac{t_i - t}{\tau_i}\vartheta^{i-1}_{\Gamma} \right\|^2_{H^{-1/2}(\Gamma)} \\ &\leq \left\|\widehat{\vartheta}_{\mathcal{H},\tau}\right\|^2_{L^2(0,T;H^{-1/2}(\Gamma))} \leq \left(1+\frac{1}{\sigma}\right)\|\vartheta_\Gamma\|^2_{L^2(0,T;H^{-1/2}(\Gamma))},\end{aligned}$$

where we have used the Propoistion 4.28 in the last inequality.

Finally we show the bound in the H^1-seminorm of $\widehat{\vartheta}_{\mathcal{H},\tau}$

$$\begin{aligned}\left\|\partial_t\widehat{\vartheta}_{\mathcal{H},\tau}\right\|^2_{L^2(0,T;L^2(\Omega))} &= \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \left\| \frac{\vartheta^i_{\mathcal{H}} - \vartheta^{i-1}_{\mathcal{H}}}{\tau_i} \right\|^2_{L^2(\Omega)} \leq \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \left\| \frac{\overline{\vartheta}^i_{\Gamma} - \overline{\vartheta}^{i-1}_{\Gamma}}{\tau_i} \right\|^2_{H^{-1/2}(\Gamma)} \\ &= \left\|\partial_t\widehat{\overline{\vartheta}}_{\Gamma,\tau}\right\|^2_{L^2(0,T;H^{-1/2}(\Gamma))} \leq \left(1+\frac{1}{\sigma}\right)\|\partial_t\vartheta_\Gamma\|^2_{L^2(0,T;H^{-1/2}(\Gamma))}.\end{aligned}$$

where again the last estimate is a conclusion from Proposition 4.28. □

Finally, we also need some norm estimates of $\vartheta_{\mathcal{H},\tau}$ in connection with the functions β_τ, β^i_ε, ln and Ln_ε, which are provided by the following lemma.

Proposition 4.37. *We have for a constant $c > 0$*

$$\left\|\mathrm{Ln}_\varepsilon \vartheta^i_{\mathcal{H}}\right\|_{L^\infty(\Omega)\cap H^1(\Omega)} \leq c \tag{4.40a}$$

$$\left\|\beta^i_\varepsilon(\vartheta^i_{\mathcal{H}})\right\|_{L^\infty(\Omega)\cap H^1(\Omega)} \leq c \tag{4.40b}$$

for all $1 \leq i \leq N$ and ε small enough as well as for a partition with $\tau > 0$

$$\|\beta_\tau(\vartheta_{\mathcal{H},\tau})\|_{L^\infty(Q)} \leq c \tag{4.40c}$$

$$\left\|\partial_t\widehat{\beta}_\tau(\vartheta_{\mathcal{H},\tau})\right\|_{L^2(Q)} \leq c. \tag{4.40d}$$

Proof. The estimate (4.39a) and the estimate (4.9d) of the Proposition 4.15 imply that $\ell_* \leq \ln_\varepsilon \vartheta^i_{\mathcal{H}} \leq \ell^*$ a.e. in Ω, with the notation of the Proposition. Thus also $\mathrm{Ln}_\varepsilon \vartheta^i_{\mathcal{H}}$ is bounded. For the H^1-norm we argue in the same way, but now using the estimates (4.39a) and (4.9d).

The L^∞ bound of β^i_ε is again just a consequence of (4.39a), (3.3) and (4.33a). For the estimate regarding the space derivative we have

$$\begin{aligned}\left\|\nabla\beta^i_\varepsilon(\vartheta^i_{\mathcal{H}})\right\|_{L^2(\Omega)} &\leq \left\|\beta^i_{\varepsilon,x}(\vartheta^i_{\mathcal{H}})\right\|_{L^2(\Omega)} + \left\|(\beta^i_\varepsilon)'(\vartheta^i_{\mathcal{H}})\nabla\vartheta^i_{\mathcal{H}}\right\|_{L^2(\Omega)} \\ &\leq c\left\|\beta^i_{\varepsilon,x}(\vartheta^i_{\mathcal{H}})\right\|_{L^\infty(\Omega)} + c\sup_{\Omega\times[\vartheta_*,\vartheta^*]}\left|(\beta^i_\varepsilon)'\right|\left\|\vartheta^i_{\mathcal{H}}\right\|_{H^1(\Omega)} \leq c,\end{aligned}$$

where we used the boundedness of $\left\|\beta^i_{\varepsilon,t}(\vartheta_{\mathcal{H}})\right\|_{L^\infty(\Omega)}$, provided ε is small enough due to the Proposition 4.33 and the L^∞–estimate just proved as well as the supremum estimate (4.33a) and finally (4.39a).

The next estimate is just again a consequence of (4.39a), and (3.3). For the estimate we obtain

$$\begin{aligned}
\left\|\partial_t\widehat{\beta}_\tau(\vartheta_{\mathcal{H},\tau})\right\|^2_{L^2(Q)} &\leq \sum_{i=1}^N \tau_i \left\|\frac{\beta^i(\vartheta^i_{\mathcal{H}}) - \beta^i(\vartheta^{i-1}_{\mathcal{H}})}{\tau^i}\right\|^2_{L^2(\Omega)} \\
&\leq \sum_{i=1}^N \frac{1}{\tau_i}\left(\left\|\beta^i(\vartheta^i_{\mathcal{H}}) - \beta^{i-1}(\vartheta^i_{\mathcal{H}})\right\|^2_{L^2(\Omega)} + \left\|\beta^{i-1}(\vartheta^i_{\mathcal{H}}) - \beta^i(\vartheta^{i-1}_{\mathcal{H}})\right\|^2_{L^2(\Omega)}\right), \\
&\leq \sum_{i=1}^N \frac{1}{\tau_i}\left(L_{\beta,\vartheta_*}\tau_i^2 + L^2_{\beta,\vartheta_*}\left\|\vartheta^i_{\mathcal{H}} - \vartheta^{i-1}_{\mathcal{H}}\right\|_{L^2(Q)}\right) \\
&\leq c\left(1 + \left\|\partial_t\widehat{\vartheta}_{\mathcal{H}}\right\|_{L^2(Q)}\right) \leq c,
\end{aligned}$$

where L_{β,ϑ_*} is the Lipschitz constant of β on the interval $[\vartheta_*, \vartheta^*]$, due to the boundedness of $\left\|\partial_t\widehat{\vartheta}_{\mathcal{H}}\right\|_{L^2(Q)}$ pointed out in (4.39b). □

Chapter 5

Time-Step discretization

In this chapter we want to prove the existence, uniqueness and the maximum principle for χ. Let us again point out the main equations (3.15d)-(3.15g), which have to be solved

$$\frac{\vartheta^i - \vartheta^{i-1}}{\tau_i} - G'(\chi^i)\frac{\chi^i - \chi^{i-1}}{\tau_i} - \Delta \ln \vartheta^i + \xi^i = \overline{\pi}^i(\vartheta^{i-1}), \quad \xi^i = \beta^i(\vartheta^i) \tag{5.1a}$$

$$\frac{\chi^i - \chi^{i-1}}{\tau_i} - \Delta\chi^i + F'(\chi^i) + G'(\chi^i)\vartheta^{i-1} = 0 \tag{5.1b}$$

$$\ln \vartheta^i = \ln \overline{\vartheta}^i_\Gamma \quad \text{and} \quad \partial_n \chi^i = 0 \quad \text{on } \Gamma, \quad \text{for } i = 1, \ldots, N \tag{5.1c}$$

$$\vartheta^0 = \vartheta_0 \quad \text{and} \quad \chi^0 = \chi_0 \quad \text{in } \Omega \tag{5.1d}$$

where the initial and boundary values are bounded due to the assumptions (3.6)

$$\vartheta_* \leq \overline{\vartheta}^i_\Gamma \leq \vartheta^* \quad \text{on } \Gamma, \tag{5.1e}$$

$$\vartheta_* \leq \vartheta_0 \leq \vartheta^* \quad \text{and} \quad \chi_* \leq \chi_0 \leq \chi^* \quad \text{in } \Omega \quad \text{for } i = 1, \ldots, N. \tag{5.1f}$$

Remark 5.1. (i) We point out, that the equation for ϑ^i (5.1a) is in semi-explicit form and the equation for χ^i (5.1b) is separated from the first equation (5.1a) on level i.

(ii) We observe: If there exists a solution of (5.1a), the solution satisfies $\vartheta^i > 0$ a.e. in Ω.

At this point we assume, that we have solved the discrete problem up to the time t_{i-1}. This means that there exists a triplet $(\vartheta^{i-1}, \chi^{i-1}, \xi^{i-1})$ satisfying (5.1a)-(5.1d) with i substituted by $i-1$ or being the initial value. In addition χ^{i-1} is bounded: $\chi_* \leq \chi^{i-1} \leq \chi^*$. Then, for getting to time t_i, we need three steps:

1) We get χ^i by solving (5.1b).

2) We solve a regularised version of (5.1a) resulting in some ϑ^i_ε.

3) We perform the limit $\varepsilon \to 0$ in the approximation to get the exact solution ϑ^i of (5.1a).

1. Solving for χ^i

1.1. Maximum principle

Let us fix a Lipschitz continuous function $H : \mathbb{R} \to \mathbb{R}$ of class C^1 such that

$$H(r) = 0 \quad \text{if } r \in [\chi_*, \chi^*] \quad \text{and} \quad H'(r) > 0 \quad \text{if } r \notin [\chi_*, \chi^*]. \tag{5.2a}$$

Now, we use $H(\chi)$ as a test function for (5.1b) and integrate over Ω. After integrating by parts, we obtain

$$\int_\Omega \frac{\chi^i - \chi^{i-1}}{\tau_i} H(\chi^i) + \int_\Omega \left|\nabla \chi^i\right|^2 H'(\chi^i) + \int_\Omega F'(\chi^i) H(\chi^i) + \int_\Omega G'(\chi') \vartheta^{i-1} H(\chi^i) = 0 \tag{5.2b}$$

The second integral is obviously nonnegative. Also the third and the fourth are nonnegative because F' respectively G' and H are having the same sign due to (3.2c) and (5.2a) as well as ϑ^{i-1} is nonnegative. For the first integral we have $\chi^{i-1} \in [\chi_*, \chi^*]$ by assumption. Adding the zero $-\frac{\chi^i - \chi^{i-1}}{\tau_i} H(\chi^{i-1})$ leads to

$$\int_\Omega \frac{\chi^i - \chi^{i-1}}{\tau_i} H(\chi^i) = \int_\Omega \frac{\chi^i - \chi^{i-1}}{\tau_i} (H(\chi^i) - H(\chi^{i-1})) \geq 0,$$

because H is monotone. We have deduced, that also the first integral is nonnegative. Finally, all integrals are identical zero and by definition (5.2a) of H we obtain

$$\chi_* \leq \chi^i \leq \chi^* \quad \text{a.e. in } \Omega. \tag{5.2c}$$

1.2. Existence

We prove the existence of the solution to problem (5.1b) with boundary conditions given by (5.1c). We consider the map

$$\Phi : L^2(\Omega) \to L^2(\Omega) \quad \text{and} \quad \chi^i = \Phi(\widetilde{\chi}^i) \quad \text{solving} \tag{5.3a}$$

$$\frac{\chi^i}{\tau_i} - \Delta \chi^i = \frac{\chi^{i-1}}{\tau_i} - F'(\widetilde{\chi}^i) - G'(\widetilde{\chi}^i)\vartheta^{i-1} \quad \text{in } \Omega \tag{5.3b}$$

$$\partial_n \chi^i = 0 \quad \text{on } \Gamma \tag{5.3c}$$

Then, from the theory of elliptic equations we have a unique solution satisfying

$$\left\|\Phi(\widetilde{\chi}^i)\right\|_{H^2(\Omega)} \leq C \left(\frac{1}{\tau_i} \left\|\chi^{i-1}\right\|_{L^2(\Omega)} + L_F + L_G \left\|\vartheta^{i-1}\right\|_{L^2(\Omega)} \right) \leq C, \tag{5.3d}$$

due to the regularity of Γ and the boundedness of F' and G' remarked in (3.9). In addition C is independent of $\widetilde{\chi}^i$. Thus, the range of Φ is a compact convex subset of $L^2(\Omega)$.

Now, we take a sequence $\widetilde{\chi}^i_n \to \widetilde{\chi}^i$ in $L^2(\Omega)$, then also $\Phi(\widetilde{\chi}^i_n) \in H^2_n(\Omega)$ and at least there exists a subsequence $\{n_k\}$ with

$$\widetilde{\chi}^i_{n_k} \to \widetilde{\chi}^i \quad \text{a.e. in } \Omega.$$

As a consequence for this subsequence $\{n_k\}$ we also get the convergence of the right-hand side a.e. in Ω and also in $L^2(\Omega)$ by Lebesgue dominated convergence theorem. From this follows

$$\Phi(\widetilde{\mathcal{X}}^i_{n_k}) \to \Phi(\widetilde{\mathcal{X}}^i) \quad \text{in } H^2(\Omega)$$

and by contradiction this is valid for every subsequence and so

$$\Phi(\widetilde{\mathcal{X}}^i_n) \to \Phi(\widetilde{\mathcal{X}}^i) \quad \text{in } H^2(\Omega). \tag{5.3e}$$

Therefore, Φ is also a continuous operator. So we can conclude with the Schauder fixed point Theorem B.6, that at least one $\widetilde{\mathcal{X}}^i \in H^2_n(\Omega)$ with $\mathcal{X}^i = \Phi(\mathcal{X}^i)$ does exist. Hence this yields a solution of (5.1b).

1.3. Uniqueness

We assume that the two functions $\mathcal{X}^i_1$ and $\mathcal{X}^i_2$ are fixed points of Φ. So testing the difference of the fixed point equations (5.3b) for $\mathcal{X}^i_1$ and $\mathcal{X}^i_2$ by $\mathcal{X}^i_1 - \mathcal{X}^i_2$ yields

$$\frac{1}{\tau_i}\left\|\mathcal{X}^i_1 - \mathcal{X}^i_2\right\|^2_{L^2(\Omega)} + \left\|\nabla\left(\mathcal{X}^i_1 - \mathcal{X}^i_2\right)\right\|^2_{L^2(\Omega)} \le L_F \left\|\mathcal{X}^i_1 - \mathcal{X}^i_2\right\|^2_{L^2(\Omega)} + L_G \int_\Omega \left|\vartheta^{i-1}\right| \left|\mathcal{X}^i_1 - \mathcal{X}^i_2\right|^2,$$

where we have integrated by parts and used the Lipschitz continuity of F and G. For the last integral we firstly use the generalised Hölder Inequality pointed out in Lemma B.5 with the exponents $p_1 = 2$ and $p_2 = p_3 = 4$

$$\int_\omega \left|\vartheta^{i-1}\right| \left|\mathcal{X}^i_1 - \mathcal{X}^i_2\right|^2 \le \left\|\vartheta^{i-1}\right\|_{L^2(\Omega)} \left\|\mathcal{X}^i_1 - \mathcal{X}^i_2\right\|^2_{L^4(\Omega)}.$$

To control the norm in $L^4(\Omega)$ we use the Lemma B.13. Therefore, we observe that

$$H^1(\Omega) \subset L^4(\Omega) \subset L^2(\Omega),$$

where the first embedding is compact due to the Sobolev embedding Theorem B.8 respectively for our case the Remark B.9. Then for all $\eta > 0$ exists a constant c_η such that

$$\left\|\mathcal{X}^i_1 - \mathcal{X}^i_2\right\|^2_{L^4(\Omega)} \le \eta \left\|\nabla\left(\mathcal{X}^i_1 - \mathcal{X}^i_2\right)\right\|^2_{L^2(\Omega)} + c_\eta \left\|\mathcal{X}^i_1 - \mathcal{X}^i_2\right\|^2_{L^2(\Omega)}, \tag{5.4a}$$

where we have used the seminorm in H^1 due to $\mathcal{X}^i_1 - \mathcal{X}^i_2$ fulfils homogeneous Neumann conditions. Setting $C := L_G \left\|\vartheta^{i-1}\right\|_{L^2(\Omega)}$ and combining this estimates holds

$$\left(\frac{1}{\tau_i} - L_F - c_\eta C\right)\left\|\mathcal{X}^i_1 - \mathcal{X}^i_2\right\|^2_{L^2(\Omega)} + (1 - \eta C)\left\|\nabla\left(\mathcal{X}^i_1 - \mathcal{X}^i_2\right)\right\|^2_{L^2(\Omega)} \le 0 \tag{5.4b}$$

Now, by choosing $\eta < \frac{1}{C}$ and $\tau_i < \frac{1}{L_F + c_\eta C}$ uniqueness is proved.

2. Solving for ϑ^i

2.1. Uniqueness

Firstly, we show that the uniqueness of ϑ^i is immediately following from the existence. Therefore, we test the difference of the equations (5.1a) for two solutions ϑ_1^i and ϑ_2^i by $\ln \vartheta_1^i - \ln \vartheta_2^i$ and obtain by partial integration

$$\int_\Omega \frac{\vartheta_1^i - \vartheta_2^i}{\tau_i} \left(\ln \vartheta_1^i - \ln \vartheta_2^i\right) + \int_\Omega \left|\nabla \left(\ln \vartheta_1^i - \ln \vartheta_2^i\right)\right|^2 + \int_\Omega \left(\beta^i(\vartheta_1^i) - \beta^i(\vartheta_2^i)\right) \left(\ln \vartheta_1^i - \ln \vartheta_2^i\right) = 0.$$

Obviously, the second integral is nonnegative and also the the third one by monotonicity, hence $\vartheta_1^i = \vartheta_2^i$ a.e. as the logarithm is strictly monotone.

2.2. Regularised problem

To get the existence of ϑ^i we regularise the problem. Therefore, we substitute β^i with the approximation β_ε^i introduced in (4.30b) and the ln by the regulated Ln_ε introduced in definition 4.13. Then our problem becomes

$$\frac{\vartheta_\varepsilon^i}{\tau_i} - \Delta \,\mathrm{Ln}_\varepsilon \vartheta_\varepsilon^i + \beta_\varepsilon^i(\vartheta_\varepsilon^i) = h^i \tag{5.5a}$$

$$\vartheta_\varepsilon^i = \overline{\vartheta}_\Gamma^i \quad \text{on } \Gamma, \quad \text{for } i = 1, \dots, N \tag{5.5b}$$

$$\vartheta^0 = \vartheta_0 \quad \text{in } \Omega \tag{5.5c}$$

Where h^i is defined by

$$h^i := \frac{\vartheta^{i-1}}{\tau_i} + G'(\chi^i)\frac{\chi^i - \chi^{i-1}}{\tau_i} + \overline{\pi}^i(\vartheta^{i-1}). \tag{5.5d}$$

By assumptions on G, π and the already stated solution χ^i follows $h^i \in L^2(\Omega)$.

2.3. Existence and uniqueness for $\varepsilon > 0$

We set $u_\varepsilon^i := \mathrm{Ln}_\varepsilon \vartheta_\varepsilon^i - u_\mathcal{H}^i$, where $u_\mathcal{H}^i$ solves

$$-\Delta u_\mathcal{H}^i = 0 \quad \text{in } \Omega \text{ and } \quad u_\mathcal{H}^i = \mathrm{Ln}_\varepsilon \vartheta_\Gamma^i \quad \text{on } \Gamma, \tag{5.6a}$$

and therefore is called the harmonic extension of $\mathrm{Ln}_\varepsilon \vartheta_\Gamma^i$. By the maximum principle for harmonic functions and the assumptions (3.6a) on ϑ_Γ follows

$$\mathrm{Ln}_\varepsilon \vartheta_* \le u_\mathcal{H}^i \le \mathrm{Ln}_\varepsilon \vartheta^* \quad \text{a.e. in } \Omega. \tag{5.6b}$$

Then the associated problem for u_ε^i is

$$\frac{\mathrm{Ln}_\varepsilon^{-1}\left(u_\varepsilon^i + u_\mathcal{H}^i\right)}{\tau_i} - \Delta u_\varepsilon^i + \beta_\varepsilon^i\left(\mathrm{Ln}_\varepsilon^{-1}\left(u_\varepsilon^i + u_\mathcal{H}^i\right)\right) = h^i \qquad \text{in } \Omega \tag{5.7a}$$

$$u_\varepsilon^i = 0 \qquad \text{on } \Gamma \tag{5.7b}$$

We define the operator $M : L^2(\Omega) \to L^2(\Omega)$ with

$$M(\cdot) := \frac{1}{\tau_i} \operatorname{Ln}_\varepsilon^{-1}\left(\cdot + u^i_{\mathcal{H}}\right) + \beta^i_\varepsilon\left(\operatorname{Ln}_\varepsilon^{-1}\left(\cdot + u^i_{\mathcal{H}}\right)\right).$$

We observe that M is monotone as $\operatorname{Ln}_\varepsilon^{-1}$ and β_ε are monotone. In addition M is also hemicontinuous due to $\operatorname{Ln}_\varepsilon^{-1}$ and β^i_ε are continuous.

Further, we also define $L : L^2(\Omega) \to L^2(\Omega)$ with $L = -\Delta$. Then L is a maximal monotone operator with domain $\mathcal{D}(L) = H^1_0(\Omega) \cap H^2(\Omega) \subset L^2(\Omega)$.

Remarking that with the monotonicity of M follows $(M(u) - M(\mathbf{0}), u) \geq 0$, we prove the coercivity of $L + M$. Therefore, we take $u \in \mathcal{D}(L)$, then

$$(Lu, u) + (M(u), u) = (\nabla u, \nabla u) + (M(u) - M(\mathbf{0}), u) + (M(\mathbf{0}), u) \geq \|\nabla u\|^2_{L^2(\Omega)} - C \|u\|_{L^2(\Omega)},$$

where $\mathbf{0} \in L^2(\Omega)$ is the map identical 0 and C is the following constant

$$\|M(0)\|_{L^2(\Omega)} \leq \left\|\operatorname{Ln}_\varepsilon^{-1}(u^i_{\mathcal{H}})\right\|_{L^2(\Omega)} + \left\|\beta^i_\varepsilon(\operatorname{Ln}_\varepsilon^{-1}(u^i_{\mathcal{H}}))\right\|_{L^2(\Omega)} =: C.$$

Note that the right hand side is bounded due to (5.6b), property (4.9d) and Proposition 4.37. At the end, we conclude with the Poincaré inequality, where we take a series $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(L)$ with $\lim_{n\to\infty} \|u_n\|_{L^2(\Omega)} = \infty$, then

$$\lim_{n\to\infty} \frac{(Lu_n, u_n) + (M(u_n), u_n)}{\|u_n\|_{L^2(\Omega)}} \geq \lim_{n\to\infty} \frac{\|\nabla u_n\|^2_{L^2(\Omega)}}{M_\Omega \|\nabla u_n\|_{L^2(\Omega)}} - C \geq \lim_{n\to\infty} \frac{\|u_n\|_{L^2(\Omega)}}{M_\Omega^2} - C = +\infty,$$

hence $L + M$ is coercive.

Now, we can apply Corollary 4.7 and get that $L+M$ is a maximal monotone operator with

$$\mathcal{R}(L + M) = L^2(\Omega).$$

Consequently exists a solution $u^i_\varepsilon \in H^1_0(\Omega)$ for (5.7a). But this leads to a solution $\vartheta^i_\varepsilon \in H^1(\Omega)$, as $u^i_{\mathcal{H}} \in H^2(\Omega)$ and $\operatorname{Ln}_\varepsilon^{-1}(u^i_\varepsilon) \in H^1_0(\Omega)$ for fixed $\varepsilon > 0$.

Uniqueness follows in the same way already shown in Section 2.1, because $\operatorname{Ln}_\varepsilon$ is also a strict monotone function by property (4.9c).

2.4. A priori estimates

For getting to the limit as $\varepsilon \to 0$, we first of all need some estimates uniform in ε. We choose appropriated test functions to derive some norm estimates.

In addition we remark that we use c as a general constant depending on τ^i and on several norms of $\vartheta^i_{\mathcal{H}}$ listed in the Definition 4.35 respectively the Proposition 4.37. We list only the exact values where they are important for i.e. balancing terms.

First a priori estimate

We test the equation for ϑ_ε^i (5.5a) by $\vartheta_\varepsilon^i - \vartheta_\mathcal{H}^i + \left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right)$. We remark that the harmonic extension $\vartheta_\mathcal{H}^i$ allows us to integrate by parts without an integral over the boundary, due to $\vartheta_\varepsilon^i - \vartheta_\mathcal{H}^i$ fulfils homogene Dirichlet boundary conditions. Hence the term involving the Laplacian becomes

$$\int_\Omega \Delta(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i)(\vartheta_\varepsilon^i - \vartheta_\mathcal{H}^i) = \int_\Omega \nabla(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i) \cdot \nabla(\vartheta_\varepsilon^i - \vartheta_\mathcal{H}^i).$$

After integrating we add some integrals involving $\vartheta_\mathcal{H}^i$ on both sides and use the above pointed out partial integration, which results in

$$\begin{aligned}
&\frac{1}{\tau_i}\int_\Omega \left(\vartheta_\varepsilon^i - \vartheta_\mathcal{H}^i\right)^2 + \frac{1}{\tau_i}\int_\Omega \left(\vartheta_\varepsilon^i - \vartheta_\mathcal{H}^i\right)\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right) \\
&\quad + \int_\Omega \nabla\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i\right)\cdot\nabla\vartheta_\varepsilon^i + \int_\Omega \left|\nabla\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right)\right|^2 \\
&\quad + \int_\Omega \left(\beta_\varepsilon^i(\vartheta_\varepsilon^i) - \beta_\varepsilon^i(\vartheta_\mathcal{H}^i)\right)\left(\left(\vartheta_\varepsilon^i - \vartheta_\mathcal{H}^i\right) + \left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right)\right) \\
&= -\frac{1}{\tau_i}\int_\Omega \vartheta_\mathcal{H}^i\left(\vartheta_\varepsilon^i - \vartheta_\mathcal{H}^i\right) - \frac{1}{\tau_i}\int_\Omega \vartheta_\mathcal{H}^i\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right) \\
&\quad + \int_\Omega \nabla\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i\right)\cdot\nabla\vartheta_\mathcal{H}^i - \int_\Omega \nabla\left(\operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right)\cdot\nabla\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right) \\
&\quad + \int_\Omega \left(h^i - \beta_\varepsilon^i(\vartheta_\mathcal{H}^i)\right)\left(\left(\vartheta_\varepsilon^i - \vartheta_\mathcal{H}^i\right) + \left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right)\right).
\end{aligned} \tag{5.8}$$

All integrals on the left-hand side are nonnegative due to monotonicity of $\operatorname{Ln}_\varepsilon$ and β_ε. Now, let us estimate the right-hand side. The first integral becomes, with the help of the Young Inequality B.3,

$$-\frac{1}{\tau_i}\int_\Omega \vartheta_\mathcal{H}^i\left(\vartheta_\varepsilon^i - \vartheta_\mathcal{H}^i\right) \le \frac{1}{\tau_i}\int_\Omega \left|\vartheta_\mathcal{H}^i\right|^2 + \frac{1}{4\tau_i}\int_\Omega \left(\vartheta_\varepsilon^i - \vartheta_\mathcal{H}^i\right)^2.$$

For the second one we additionally use the Poincaré inequality

$$\begin{aligned}
-\frac{1}{\tau_i}\int_\Omega \vartheta_\mathcal{H}^i\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right) &\le \frac{2M_\Omega^2}{(\tau_i)^2}\int_\Omega \left|\vartheta_\mathcal{H}^i\right|^2 + \frac{1}{8M_\Omega^2}\int_\Omega \left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right)^2 \\
&\le c\int_\Omega \left|\vartheta_\mathcal{H}^i\right|^2 + \frac{1}{8}\int_\Omega \left|\nabla\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right)\right|^2.
\end{aligned}$$

For the third and fourth we choose constants to make up the balance between the term $\int_\Omega \left|\nabla\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right)\right|^2$ on the right- and left-hand side.

$$\begin{aligned}
\int_\Omega \nabla\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i\right)\cdot\nabla\vartheta_\mathcal{H}^i &= \int_\Omega \nabla\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right)\cdot\nabla\vartheta_\mathcal{H}^i + \int_\Omega \nabla\left(\operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right)\cdot\nabla\vartheta_\mathcal{H}^i \\
&\le \frac{1}{8}\int_\Omega \left|\nabla\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right)\right|^2 + c\int_\Omega \left|\nabla\vartheta_\mathcal{H}^i\right|^2 + c\int_\Omega \left|\nabla \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right|^2.
\end{aligned}$$

Similarly, there holds

$$-\int_\Omega \nabla \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i \cdot \nabla\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right) \le \frac{1}{8}\int_\Omega \left|\nabla\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right)\right|^2 + c\int \left|\nabla \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right|^2.$$

Finally, we conclude with the last integral, again using the Poincaré inequality

$$\begin{aligned}
&\int_\Omega \left(h^i - \beta^i_\varepsilon(\vartheta^i_{\mathcal{H}})\right)\left((\vartheta^i_\varepsilon - \vartheta^i_{\mathcal{H}}) + \left(\mathrm{Ln}_\varepsilon\,\vartheta^i_\varepsilon - \mathrm{Ln}_\varepsilon\,\vartheta^i_{\mathcal{H}}\right)\right)\\
&\quad\le \left(\tau_i + 2M_\Omega^2\right)\int_\Omega \left(h^i - \beta^i_\varepsilon(\vartheta^i_{\mathcal{H}})\right)^2 + \frac{1}{4\tau_i}\int_\Omega \left(\vartheta^i_\varepsilon - \vartheta^i_{\mathcal{H}}\right)^2 + \frac{1}{8M_\Omega^2}\int_\Omega \left(\mathrm{Ln}_\varepsilon\,\vartheta^i_\varepsilon - \mathrm{Ln}_\varepsilon\,\vartheta^i_{\mathcal{H}}\right)^2\\
&\quad\le c\int_\Omega \left(h^i - \beta^i_\varepsilon(\vartheta^i_{\mathcal{H}})\right)^2 + \frac{1}{4\tau_i}\int_\Omega \left(\vartheta^i_\varepsilon - \vartheta^i_{\mathcal{H}}\right)^2 + \frac{1}{8}\int_\Omega \left|\nabla\left(\mathrm{Ln}_\varepsilon\,\vartheta^i_\varepsilon - \mathrm{Ln}_\varepsilon\,\vartheta^i_{\mathcal{H}}\right)\right|^2
\end{aligned}$$

Combining all these inequalities and taking the Definition 4.35 of $\vartheta^i_{\mathcal{H}}$ as well as the first two inequalities of Proposition 4.37 into account leads to

$$\left\|\vartheta^i_\varepsilon - \vartheta^i_{\mathcal{H}}\right\|^2_{L^2(\Omega)} + \left\|\nabla\left(\mathrm{Ln}_\varepsilon\,\vartheta^i_\varepsilon - \mathrm{Ln}_\varepsilon\,\vartheta^i_{\mathcal{H}}\right)\right\|^2_{L^2(\Omega)} + \left\|\left(\mathrm{Ln}'_\varepsilon\,\vartheta^i_\varepsilon\right)^{1/2}\nabla\vartheta^i_\varepsilon\right\|_{L^2(\Omega)} \le c$$

Finally, using the Poincaré inequality yields

$$\left\|\vartheta^i_\varepsilon\right\|_{L^2(\Omega)} + \left\|\mathrm{Ln}_\varepsilon\,\vartheta^i_\varepsilon\right\|_{H^1(\Omega)} + \left\|\left(\mathrm{Ln}'_\varepsilon\,\vartheta^i_\varepsilon\right)^{1/2}\nabla\vartheta^i_\varepsilon\right\|_{L^2(\Omega)} \le c \tag{5.10}$$

A consequence We set $\Omega^-_{i,\varepsilon} := \left\{x \in \Omega :\ \vartheta^i_\varepsilon(x) \le 1\right\}$ and $\Omega^+_{i,\varepsilon} = \left\{x \in \Omega :\ \vartheta^i_\varepsilon(x) > 1\right\}$. Then with inequalities (4.9c) yields

$$\begin{aligned}
&\int_\Omega \mathrm{Ln}'_\varepsilon(\vartheta^i_\varepsilon)\left|\nabla\vartheta^i_\varepsilon\right|^2 \ge \int_{\Omega^-_{i,\varepsilon}} \mathrm{Ln}'_\varepsilon(\vartheta^i_\varepsilon)\left|\nabla\vartheta^i_\varepsilon\right|^2 \ge \int_{\Omega^-_{i,\varepsilon}} \left|\nabla\vartheta^i_\varepsilon\right|^2\\
&\int_\Omega \mathrm{Ln}'_\varepsilon(\vartheta^i_\varepsilon)\left|\nabla\vartheta^i_\varepsilon\right|^2 \ge \int_{\Omega^+_{i,\varepsilon}} \mathrm{Ln}'_\varepsilon(\vartheta^i_\varepsilon)\left|\nabla\vartheta^i_\varepsilon\right|^2 \ge \int_{\Omega^+_{i,\varepsilon}} \frac{\left|\nabla\vartheta^i_\varepsilon\right|^2}{\vartheta^i_\varepsilon}.
\end{aligned} \tag{5.11a}$$

Remarking that $\frac{|\nabla\vartheta^i_\varepsilon|^2}{\vartheta^i_\varepsilon} = \left|\nabla(\vartheta^i)^{1/2}_\varepsilon\right|^2$ and using the norm estimates (5.10) implies

$$\int_{\Omega^-_{i,\varepsilon}} \left|\nabla\vartheta^i_\varepsilon\right|^2 \le c \quad\text{and}\quad \int_{\Omega^+_{i,\varepsilon}} \left|\nabla(\vartheta^i_\varepsilon)^{1/2}\right|^2 \le c. \tag{5.11b}$$

With the help of these results we can estimate $\nabla\vartheta^i_\varepsilon$ in a suitable norm. Accounting for the first estimate in (5.11b), we observe

$$\|\nabla\vartheta\|_{L^{4/3}(\Omega^-_{i,\varepsilon})} \le c\,\|\nabla\vartheta\|_{L^2(\Omega^-_{i,\varepsilon})} \le c. \tag{5.11c}$$

On the other hand using $\nabla\vartheta^i_\varepsilon = 2(\vartheta^i_\varepsilon)^{1/2}\nabla(\vartheta^i_\varepsilon)^{1/2}$ and the Hölder Inequality B.5 with the exponents $1/4 + 1/2 = 3/4$ we obtain

$$\|\nabla\vartheta\|_{L^{4/3}(\Omega^+_{i,\varepsilon})} \le 2\left\|(\vartheta^i_\varepsilon)^{1/2}\right\|_{L^4(\Omega^+_{i,\varepsilon})}\left\|\nabla(\vartheta^i_\varepsilon)^{1/2}\right\|_{L^2(\Omega^+_{i,\varepsilon})} \le 2\left\|\vartheta^i_\varepsilon\right\|^{1/2}_{L^2(\Omega)}\left\|\nabla(\vartheta^i_\varepsilon)^{1/2}\right\|_{L^2(\Omega^+_{i,\varepsilon})} \le c \tag{5.11d}$$

by the estimates (5.10) and (5.11b).

Finally, we observe

$$\begin{aligned}
&\left\|\nabla\vartheta^i_\varepsilon\right\|^{4/3}_{L^{4/3}(\Omega)} = \left\|\nabla\vartheta^i_\varepsilon\right\|^{4/3}_{L^{4/3}(\Omega^-_{i,\varepsilon})} + \left\|\nabla\vartheta^i_\varepsilon\right\|^{4/3}_{L^{4/3}(\Omega^+_{i,\varepsilon})} \quad\text{whence}\\
&\left\|\nabla\vartheta^i_\varepsilon\right\|^{4/3}_{L^{4/3}(\Omega)} \le c\left(\left\|\nabla\vartheta^i_\varepsilon\right\|^2_{L^{4/3}(\Omega^-_{i,\varepsilon})} + \left\|\nabla\vartheta^i_\varepsilon\right\|^2_{L^{4/3}(\Omega^+_{i,\varepsilon})}\right)
\end{aligned}$$

so that using (5.11c) and (5.11d), we conclude with

$$\left\|\nabla\vartheta_\varepsilon^i\right\|_{L^{4/3}(\Omega)} \le c. \tag{5.12}$$

Second a priori estimate

At first we remind of the product rule and the notation for β_ε^i and its derivatives

$$\nabla\beta_\varepsilon^i(\vartheta_\varepsilon^i) = {\beta_\varepsilon^i}'(\vartheta_\varepsilon^i)\nabla\vartheta_\varepsilon^i + \beta_{\varepsilon,x}^i(\vartheta_\varepsilon^i).$$

We again test (5.5a) but now with $\beta_\varepsilon^i(\vartheta_\varepsilon^i) - \beta_\varepsilon^i(\vartheta_\mathcal{H}^i)$. Once more we can integrate by parts without a remainder. We point out the term involving the Laplacian

$$\int_\Omega \Delta(\mathrm{Ln}_\varepsilon\,\vartheta_\varepsilon^i)(\beta_\varepsilon^i(\vartheta_\varepsilon^i) - \beta_\varepsilon^i(\vartheta_\mathcal{H}^i)) = \int_\Omega \nabla(\mathrm{Ln}_\varepsilon\,\vartheta_\varepsilon^i)\cdot({\beta_\varepsilon^i}'(\vartheta_\varepsilon^i)\nabla\vartheta_\varepsilon^i + \beta_{\varepsilon,x}^i(\vartheta_\varepsilon^i) + \nabla\beta_\varepsilon^i(\vartheta_\mathcal{H}^i)).$$

Integrating and using the above stated formula results in

$$\begin{aligned}
&\frac{1}{\tau_i}\int_\Omega \left(\vartheta_\varepsilon^i - \vartheta_\mathcal{H}^i\right)\left(\beta_\varepsilon^i(\vartheta_\varepsilon^i) - \beta_\varepsilon^i(\vartheta_\mathcal{H}^i)\right) + \int_\Omega \nabla\left(\mathrm{Ln}_\varepsilon\,\vartheta_\varepsilon^i\right)\cdot{\beta_\varepsilon^i}'(\vartheta_\varepsilon^i)\nabla\vartheta_\varepsilon^i + \int_\Omega \left(\beta_\varepsilon^i(\vartheta_\varepsilon^i) - \beta_\varepsilon^i(\vartheta_\mathcal{H}^i)\right)^2 \\
&\quad = -\frac{1}{\tau_i}\int_\Omega \vartheta_\mathcal{H}^i\left(\beta_\varepsilon^i(\vartheta_\varepsilon^i) - \beta_\varepsilon^i(\vartheta_\mathcal{H}^i)\right) - \int_\Omega \nabla\,\mathrm{Ln}_\varepsilon\,\vartheta_\varepsilon^i\cdot\beta_{\varepsilon,x}^i(\vartheta_\varepsilon^i) + \int_\Omega \nabla\left(\mathrm{Ln}_\varepsilon\,\vartheta_\varepsilon^i\right)\cdot\nabla\beta_\varepsilon^i(\vartheta_\mathcal{H}^i) \\
&\qquad + \int_\Omega \left(h^i - \beta_\varepsilon^i(\vartheta_\mathcal{H}^i)\right)\left(\beta_\varepsilon^i(\vartheta_\varepsilon^i) - \beta_\varepsilon^i(\vartheta_\mathcal{H}^i)\right)
\end{aligned}$$

All integrals on the left-hand side are again nonnegative. For the right-hand side we estimate term by term. For the first one it yields

$$\frac{1}{\tau_i}\int_\Omega \vartheta_\mathcal{H}^i\left(\beta_\varepsilon^i(\vartheta_\varepsilon^i) - \beta_\varepsilon^i(\vartheta_\mathcal{H}^i)\right) \le c\left\|\vartheta_\mathcal{H}^i\right\|_{L^2(\Omega)}^2 + \frac{1}{8}\int_\Omega \left(\beta_\varepsilon^i(\vartheta_\varepsilon^i) - \beta_\varepsilon^i(\vartheta_\mathcal{H}^i)\right)^2,$$

where we stress out that c now depends also on τ_i. In the next integral we use the Proposition 4.33

$$\begin{aligned}
\int_\Omega \nabla\,\mathrm{Ln}_\varepsilon\,\vartheta_\varepsilon^i\cdot\beta_{\varepsilon,x}^i(\vartheta_\varepsilon^i) &\le \widetilde{c}_\delta\left\|\mathrm{Ln}_\varepsilon\,\vartheta_\varepsilon^i\right\|_{H^1(\Omega)}^2 + \widetilde{\delta}\int_\Omega \left(1 + \beta_\varepsilon^i(\vartheta_\varepsilon^i)\right)^2 \\
&\le c_\delta\left\|\mathrm{Ln}_\varepsilon\,\vartheta_\varepsilon^i\right\|_{H^1(\Omega)}^2 + \delta\int_\Omega \left(\beta_\varepsilon^i(\vartheta_\varepsilon^i) - \beta_\varepsilon^i(\vartheta_\mathcal{H}^i)\right)^2 + c.
\end{aligned}$$

The following integral we estimate by using again the Young Inequality B.3

$$\int_\Omega \nabla\left(\mathrm{Ln}_\varepsilon\,\vartheta_\varepsilon^i\right)\cdot\nabla\beta_\varepsilon^i(\vartheta_\mathcal{H}^i) \le \frac{1}{2}\left\|\mathrm{Ln}_\varepsilon\,\vartheta_\varepsilon^i\right\|_{H^1(\Omega)}^2 + \frac{1}{2}\left\|\beta_\varepsilon^i(\vartheta_\mathcal{H}^i)\right\|_{H^1(\Omega)}^2.$$

Finally, we have for the last integral achieved

$$\int_\Omega \left(h^i - \beta_\varepsilon^i(\vartheta_\mathcal{H}^i)\right)\left(\beta_\varepsilon^i(\vartheta_\varepsilon^i) - \beta_\varepsilon^i(\vartheta_\mathcal{H}^i)\right) \le c\left\|h^i - \beta_\varepsilon^i(\vartheta_\mathcal{H}^i)\right\|_{L^2(\Omega)}^2 + \frac{1}{8}\int_\Omega \left(\beta_\varepsilon^i(\vartheta_\varepsilon^i) - \beta_\varepsilon^i(\vartheta_\mathcal{H}^i)\right)^2.$$

Combining all these inequalities and choosing $\delta = 1/4$ leads to

$$\frac{1}{2}\left\|\beta_\varepsilon^i(\vartheta_\varepsilon^i) - \beta_\varepsilon^i(\vartheta_\mathcal{H}^i)\right\|_{L^2(\Omega)}^2 \le c\left\|\vartheta_\mathcal{H}^i\right\|_{L^2(\Omega)} + c\left\|\mathrm{Ln}_\varepsilon\,\vartheta_\varepsilon^i\right\|_{H^1(\Omega)}^2 + c\left\|\beta_\varepsilon^i(\vartheta_\mathcal{H}^i)\right\|_{H^1(\Omega)}^2 + c\left\|h^i\right\|_{L^2(\Omega)}^2 + c.$$

Taking the Proposition 4.37 as well as the norm estimate (5.10) into account we conclude with

$$\left\|\beta_\varepsilon^i(\vartheta_\varepsilon^i)\right\|_{L^2(\Omega)} \le c. \tag{5.14}$$

2.5. Passing to the limit in ε

The following convergences hold

$$\vartheta^i_\varepsilon \rightharpoonup \vartheta^i \quad \text{weakly in } L^2(\Omega) \tag{5.15a}$$

$$\mathrm{Ln}_\varepsilon\, \vartheta^i_\varepsilon \rightharpoonup \ell^i \quad \text{weakly in } H^1(\Omega) \tag{5.15b}$$

$$\nabla\vartheta^i_\varepsilon \rightharpoonup \nabla\vartheta^i \quad \text{weakly in } L^{4/3}(\Omega) \tag{5.15c}$$

$$\beta^i_\varepsilon(\vartheta^i_\varepsilon) \rightharpoonup \xi^i \quad \text{weakly in } L^2(\Omega) \tag{5.15d}$$

With (5.15a) and (5.15c) we can conclude

$$\vartheta^i_\varepsilon \rightharpoonup \vartheta^i \quad \text{weak in } W^{1,4/3}(\Omega).$$

Now, using the fact that $W^{1,4/3}(\Omega)$ is compactly embedded in $L^2(\Omega)$ by the embedding theorem (B.8) and in this case (B.10) we achieve

$$\vartheta^i_\varepsilon \to \vartheta^i \quad \text{strongly in } L^2(\Omega). \tag{5.16}$$

Therewith, we obtain by (5.15b) also

$$\mathrm{Ln}_\varepsilon\, \vartheta^i_\varepsilon \to \ell^i \quad \text{strongly in } L^2(\Omega). \tag{5.17}$$

With $\mathrm{Ln}_\varepsilon\, \vartheta^i_\varepsilon = \varepsilon\vartheta^i_\varepsilon + \ln_\varepsilon \vartheta^i_\varepsilon$ and

$$\varepsilon\vartheta^i_\varepsilon \to 0 \quad \text{strongly in } L^2(\Omega)$$

yields also

$$\ln_\varepsilon \vartheta^i_\varepsilon \to \ell^i \quad \text{strongly in } L^2(\Omega). \tag{5.18}$$

But this leads to

$$\limsup_{k\to\infty} \vartheta^i_{\varepsilon_k} \ln_{\varepsilon_k} \vartheta^i_{\varepsilon_k} = \vartheta^i\ell^i$$

and by the Proposition 4.12 we can conclude

$$\ell^i \in \ln\vartheta^i \quad \text{and therefore } \vartheta^i > 0 \text{ and } \ell^i = \ln\vartheta^i. \tag{5.19}$$

Further from Lemma 4.34 follows $\xi^i = \beta^i(\vartheta^i)$. Finally, the remaining terms involving F' and G' can be identified with their limits due to the convergence given in (5.16) and the assumptions (3.2b).

Chapter 6

Convergence to continuous solution

Henceforth we have found for every uniform partition $\mathcal{P}$ with diameter $\tau > 0$ a unique triplet $(\vartheta^{\mathcal{P}}, \chi^{\mathcal{P}}, \xi^{\mathcal{P}})$ solution of the discrete problem according the Definition 3.12. To show the convergence to the continuous solution we need firstly some estimates uniform with respect to τ. Therefore, we choose certain test functions similar we have already used in Chapter 5 for the a priori estimates. The resulting norm estimates lead to weak convergences in certain spaces and by compactness and embedding results we can conclude with a strong convergence to a solution of the continuous problem.

Now take a discrete solution $(\vartheta^{\mathcal{P}}, \chi^{\mathcal{P}}, \xi^{\mathcal{P}})$ with $\xi^i = \beta^i(\vartheta^i)$ for all $i = 1, \ldots, N$ satisfying

$$\frac{\vartheta^i - \vartheta^{i-1}}{\tau_i} - G'(\chi^i)\frac{\chi^i - \chi^{i-1}}{\tau_i} - \Delta \ln \vartheta^i + \beta^i(\vartheta^i) = \overline{\pi}^i(\vartheta^{i-1}) \tag{6.1a}$$

$$\frac{\chi^i - \chi^{i-1}}{\tau_i} - \Delta\chi^i + F'(\chi^i) + G'(\chi^i)\vartheta^{i-1} = 0 \tag{6.1b}$$

for a.a. $(x,t) \in Q$ with i such that $t \in (t_{i-1}, t_i)$.

1. Estimates uniform with respect to τ

1.1. First a priori estimate

We test the equation for ϑ^i (6.1a) by $\vartheta^i - \vartheta^i_{\mathcal{H}} + \delta\left(\ln\vartheta^i - \ln\vartheta^i_{\mathcal{H}}\right)$ and add the equation for χ^i (6.1b) tested by $\frac{\chi^i - \chi^{i-1}}{\tau_i}$. We point out that the first term of (6.1a) can be rearranged as follows

$$\frac{1}{\tau^i}\int_\Omega \left(\vartheta^i - \vartheta^{i-1}\right)\left(\vartheta^i - \vartheta^i_{\mathcal{H}}\right) = \frac{1}{\tau_i}\int_\Omega \left[\frac{1}{2}\left(\vartheta^i - \vartheta^{i-1}\right)^2 + \frac{1}{2}\left(\vartheta^i - \vartheta^i_{\mathcal{H}}\right)^2 - \frac{1}{2}\left(\vartheta^{i-1} - \vartheta^i_{\mathcal{H}}\right)^2\right].$$

Now we get the following equation, where we also added some integrals related to $\vartheta_{\mathcal{H}}$ on both sides allowing us to integrate by parts without remainder

$$\begin{aligned}
&\int_\Omega \frac{1}{\tau_i}\left[\frac{1}{2}\left(\vartheta^i-\vartheta^{i-1}\right)^2+\frac{1}{2}\left(\vartheta^i-\vartheta^i_{\mathcal{H}}\right)^2-\frac{1}{2}\left(\vartheta^{i-1}-\vartheta^i_{\mathcal{H}}\right)^2\right]\\
&\quad+\frac{\delta}{\tau_i}\int_\Omega\left(\vartheta^i-\vartheta^{i-1}\right)\ln\vartheta^i+\int_\Omega\nabla\ln\vartheta^i\cdot\nabla\vartheta^i+\delta\int_\Omega\left|\nabla\left(\ln\vartheta^i-\ln\vartheta^i_{\mathcal{H}}\right)\right|^2\\
&\quad+\int_\Omega\left(\beta^i(\vartheta^i)-\beta^i(\vartheta^i_{\mathcal{H}})\right)\left(\left(\vartheta^i-\vartheta^i_{\mathcal{H}}\right)+\delta\left(\ln\vartheta^i-\ln\vartheta^i_{\mathcal{H}}\right)\right)\\
&\quad+\int_\Omega\frac{\left(\chi^i-\chi^{i-1}\right)^2}{(\tau_i)^2}+\frac{1}{\tau_i}\int_\Omega\left[\frac{1}{2}\left|\nabla\chi^i\right|^2+\frac{1}{2}\left|\nabla\left(\chi^i-\chi^{i-1}\right)\right|^2-\frac{1}{2}\left|\nabla\chi^{i-1}\right|^2\right]\\
&=\frac{\delta}{\tau_i}\int_\Omega\left(\vartheta^i-\vartheta^{i-1}\right)\ln\vartheta^i_{\mathcal{H}}+\int_\Omega\nabla\ln\vartheta^i\cdot\nabla\vartheta^i_{\mathcal{H}}-\delta\int_\Omega\nabla\ln\vartheta^i_{\mathcal{H}}\cdot\nabla\left(\ln\vartheta^i-\ln\vartheta^i_{\mathcal{H}}\right)\\
&\quad+\int_\Omega G'(\chi^i)\frac{\chi^i-\chi^{i-1}}{\tau_i}\left(\vartheta^i-\vartheta^{i-1}-\vartheta^i_{\mathcal{H}}\right)+\delta\int_\Omega G'(\chi^i)\frac{\chi^i-\chi^{i-1}}{\tau_i}\left(\ln(\vartheta^i)-\ln(\vartheta^i_{\mathcal{H}})\right)\\
&\quad+\int_\Omega\left(\overline{\pi}^i(\vartheta^{i-1})-\beta^i(\vartheta^i_{\mathcal{H}})\right)\left(\vartheta^i-\vartheta^i_{\mathcal{H}}\right)+\delta\int_\Omega\left(\overline{\pi}^i(\vartheta^{i-1})-\beta^i(\vartheta^i_{\mathcal{H}})\right)\left(\ln\vartheta^i-\ln\vartheta^i_{\mathcal{H}}\right)\\
&\quad-\int_\Omega F'(\chi^i)\frac{\chi^i-\chi^{i-1}}{\tau_i}
\end{aligned}\tag{6.2}$$

In the first line we bring the term with the minus sign on the other side and estimate

$$\frac{1}{2\tau_i}\int_\Omega\left(\vartheta^{i-1}-\vartheta^i_{\mathcal{H}}\right)^2\le\frac{1}{\tau_i}\left\|\vartheta^{i-1}-\vartheta^{i-1}_{\mathcal{H}}\right\|^2_{L^2(\Omega)}+\frac{1}{8\tau_i}\left\|\vartheta^i_{\mathcal{H}}-\vartheta^{i-1}_{\mathcal{H}}\right\|^2_{L^2(\Omega)}.$$

A lower bound have to be established in the second line. Therefore we notice the primitive of ln

$$\int_1^r\ln s\,\mathrm{d}s=r\left(\ln r-1\right)+1\quad\text{for } r>0.$$

And now observe due to the monotonicity of ln

$$\int_\Omega\left(\vartheta^i-\vartheta^{i-1}\right)\ln\vartheta^i\ge\int_\Omega\int_{\vartheta^{i-1}}^{\vartheta^i}\ln s\,\mathrm{d}s=\int_\Omega\left(\vartheta^i\left(\ln\vartheta^i-1\right)-\vartheta^{i-1}\left(\ln\vartheta^{i-1}-1\right)\right).$$

Now we follow up the right-hand side. The first integral we discretely integrate by parts

$$\begin{aligned}\frac{\delta}{\tau_i}\int_\Omega\left(\vartheta^i-\vartheta^{i-1}\right)\ln\vartheta^i_{\mathcal{H}}=\frac{\delta}{\tau_i}\int_\Omega&\big[(\vartheta^i-\vartheta^i_{\mathcal{H}})\ln\vartheta^i_{\mathcal{H}}-(\vartheta^{i-1}-\vartheta^{i-1}_{\mathcal{H}})(\ln\vartheta^i_{\mathcal{H}}-\ln\vartheta^{i-1}_{\mathcal{H}})\\
&-(\vartheta^{i-1}-\vartheta^{i-1}_{\mathcal{H}})\ln\vartheta^{i-1}_{\mathcal{H}}+(\vartheta^i_{\mathcal{H}}-\vartheta^{i-1}_{\mathcal{H}})\ln\vartheta^i_{\mathcal{H}}\big].\end{aligned}$$

For the next one, by using the Young inequality there holds

$$\begin{aligned}\int_\Omega\nabla\ln\vartheta^i\cdot\nabla\vartheta^i_{\mathcal{H}}&=\int_\Omega\nabla\left(\ln\vartheta^i-\ln\vartheta^i_{\mathcal{H}}\right)\cdot\nabla\vartheta^i_{\mathcal{H}}+\int_\Omega\nabla\ln\vartheta^i_{\mathcal{H}}\cdot\nabla\vartheta^i_{\mathcal{H}}\\
&\le\frac{\delta}{8}\int_\Omega\left|\nabla\left(\ln\vartheta^i-\ln\vartheta^i_{\mathcal{H}}\right)\right|^2+\frac{4}{\delta}\left\|\nabla\vartheta^i_{\mathcal{H}}\right\|^2_{L^2(\Omega)}+\frac{\delta}{8}\left\|\nabla\ln\vartheta^i_{\mathcal{H}}\right\|^2_{L^2(\Omega)}.\end{aligned}$$

Similarly, it follows that

$$-\delta\int_\Omega\nabla\ln\vartheta^i_{\mathcal{H}}\nabla\left(\ln\vartheta^i-\ln\vartheta^i_{\mathcal{H}}\right)\le\frac{\delta}{8}\int_\Omega\left|\nabla\left(\ln\vartheta^i-\ln\vartheta^i_{\mathcal{H}}\right)\right|^2+\frac{2}{\delta}\left\|\nabla\ln\vartheta^i_{\mathcal{H}}\right\|^2_{L^2(\Omega)}.$$

For the next both integrals in the next line we use the boundedness of G' by L_G introduced in (3.9) and get

$$\int_\Omega G'(\chi^i)\frac{\chi^i-\chi^{i-1}}{\tau_i}\left(\vartheta^i-\vartheta^{i-1}-\vartheta^i_{\mathcal{H}}\right)$$
$$\leq \frac{1}{4}\int_\Omega\left(\frac{\chi^i-\chi^{i-1}}{\tau_i}\right)^2+L_G^2\int_\Omega\left(\vartheta^i-\vartheta^{i-1}\right)^2+L_G^2\left\|\vartheta^i_{\mathcal{H}}\right\|^2_{L^2(\Omega)}.$$

The second integral can be estimated with the help of the Poincaré inequality as follows

$$\delta\int_\Omega G'(\chi^i)\frac{\chi^i-\chi^{i-1}}{\tau_i}\left(\ln(\vartheta^i)-\ln(\vartheta^i_{\mathcal{H}})\right)$$
$$\leq \frac{\delta}{8M_\Omega^2}\int_\Omega\left(\ln\vartheta^i-\ln\vartheta^i_{\mathcal{H}}\right)^2+2\delta M_\Omega^2\int_\Omega G'(\chi^i)^2\left(\frac{\chi^i-\chi^{i-1}}{\tau_i}\right)^2$$
$$\leq \frac{\delta}{8}\int_\Omega\left|\nabla\left(\ln(\vartheta^i)-\ln(\vartheta^i_{\mathcal{H}})\right)\right|^2+2\delta M_\Omega^2L_G^2\int_\Omega\left(\frac{\chi^i-\chi^{i-1}}{\tau_i}\right)^2.$$

Now, we want to remind of the assumptions on π pointed out in (3.2j). Therewith, one has

$$\int_\Omega\left(\overline{\pi}^i(\vartheta^{i-1})-\beta^i(\vartheta^i_{\mathcal{H}})\right)\left(\vartheta^i-\vartheta^i_{\mathcal{H}}\right)\leq\int_\Omega\left(\left|\beta^i(\vartheta^i_{\mathcal{H}})\right|+L_\pi\left|\vartheta^{i-1}\right|+\overline{\pi}^i_0\right)\left|\vartheta^i-\vartheta^i_{\mathcal{H}}\right|$$
$$\leq\int_\Omega\left(\vartheta^i-\vartheta^i_{\mathcal{H}}\right)^2+2L_\pi^2\int_\Omega\left(\vartheta^{i-1}-\vartheta^{i-1}_{\mathcal{H}}\right)^2+2R,$$

where $R:=\left\|\beta^i(\vartheta^i_{\mathcal{H}})\right\|^2_{L^2(\Omega)}+L_\pi^2\left\|\vartheta^{i-1}_{\mathcal{H}}\right\|^2_{L^2(\Omega)}+\left\|\overline{\pi}^i_0\right\|^2_{L^2(\Omega)}$ is a remainder term. In a similar way it yields

$$\delta\int_\Omega\left(\overline{\pi}^i(\vartheta^{i-1})-\beta^i(\vartheta^i_{\mathcal{H}})\right)\left(\ln\vartheta^i-\ln\vartheta^i_{\mathcal{H}}\right)$$
$$\leq\frac{\delta}{8M_\Omega^2}\int_\Omega\left(\ln\vartheta^i-\ln\vartheta^i_{\mathcal{H}}\right)^2+2\delta M_\Omega^2\int_\Omega\left|\overline{\pi}^i(\vartheta^i)-\beta^i(\vartheta^i_{\mathcal{H}})\right|^2$$
$$\leq\frac{\delta}{8}\int_\Omega\left|\nabla\left(\ln\vartheta^i-\ln\vartheta^i_{\mathcal{H}}\right)\right|^2+4\delta M_\Omega^2L_\pi^2\int_\Omega\left(\vartheta^{i-1}-\vartheta^{i-1}_{\mathcal{H}}\right)^2+4\delta M_\Omega^2R$$

whereby R is defined like above. Finally, the last integral becomes

$$\int_\Omega F'(\chi^i)\frac{\chi^i-\chi^{i-1}}{\tau_i}\leq\frac{1}{2}\int_\Omega\left(\frac{\chi^i-\chi^{i-1}}{\tau_i}\right)^2+\frac{L_F^2}{2},$$

due to the boundedness of F'.

Before combining all these inequalities, we put all the norms pointed out in Proposition 4.37 with respect to $\vartheta^i_{\mathcal{H}}$ and also the constants with respect to G, F as well as the bounded norm $\left\|\overline{\pi}^i_0\right\|_{L^2(\Omega)}$ in one constant $C>0$.

Now, we combine all these inequalities and get the estimate

$$
\begin{aligned}
&\left(\frac{1}{4\tau_i} - C\right) \left\|\vartheta^i - \vartheta^{i-1}\right\|^2_{L^2(\Omega)} + \left(\frac{1}{2\tau_i} - C\right) \left\|\vartheta^i - \vartheta^i_{\mathcal{H}}\right\|^2_{L^2(\Omega)} \\
&\quad + \frac{\delta}{\tau_i} \int_\Omega \big(\vartheta^i(\ln\vartheta^i - 1) - \vartheta^{i-1}(\ln\vartheta^{i-1} - 1)\big) \\
&\quad + \left\|\left(\vartheta^i\right)^{-\frac{1}{2}} \nabla\vartheta^i\right\|^2_{L^2(\Omega)} + \frac{\delta}{2} \left\|\nabla\left(\ln\vartheta^i - \ln\vartheta^i_{\mathcal{H}}\right)\right\|^2_{L^2(\Omega)} \\
&\quad + \frac{1-\delta C}{2(\tau_i)^2} \left\|\chi^i - \chi^{i-1}\right\|^2_{L^2(\Omega)} + \frac{1}{2\tau_i}\left[\left\|\nabla\chi^i\right\|^2_{L^2(\Omega)} + \left\|\nabla\left(\chi^i - \chi^{i-1}\right)\right\|^2_{L^2(\Omega)} - \left\|\nabla\chi^{i-1}\right\|^2_{L^2(\Omega)}\right] \\
&\leq \left(\frac{1}{\tau_i} + C\right) \left\|\vartheta^{i-1} - \vartheta^{i-1}_{\mathcal{H}}\right\|^2_{L^2(\Omega)} + \frac{1}{8\tau_i} \left\|\vartheta^i_{\mathcal{H}} - \vartheta^{i-1}_{\mathcal{H}}\right\|^2_{L^2(\Omega)} + C \\
&\quad + \frac{\delta}{\tau_i} \int_\Omega \left[(\vartheta^i - \vartheta^i_{\mathcal{H}})\ln\vartheta^i_{\mathcal{H}} - (\vartheta^{i-1} - \vartheta^{i-1}_{\mathcal{H}})(\ln\vartheta^i_{\mathcal{H}} - \ln\vartheta^{i-1}_{\mathcal{H}}) - (\vartheta^{i-1} - \vartheta^{i-1}_{\mathcal{H}})\ln\vartheta^{i-1}_{\mathcal{H}}\right] \\
&\quad + \frac{\delta}{\tau_i} \int_\Omega \left(\vartheta^i_{\mathcal{H}} - \vartheta^{i-1}_{\mathcal{H}}\right)\ln\vartheta^i_{\mathcal{H}}.
\end{aligned}
$$

Multiplying by τ_i and summing up from 1 to m with $1 \leq m \leq N$, choosing $\tau < \frac{1}{4C}$ and $\delta < \frac{1}{2C}$, whereby we, without loss of generality, assume $C \geq 1$ and normalising yields

$$
\begin{aligned}
&\frac{1}{4}\sum_{i=1}^m \left\|\vartheta^i - \vartheta^{i-1}\right\|^2_{L^2(\Omega)} + \frac{1}{4}\left\|\vartheta^m - \vartheta^m_{\mathcal{H}}\right\|^2_{L^2(\Omega)} + \frac{1}{4}\int_\Omega (\vartheta^m(\ln\vartheta^m - 1) + 1) \\
&\quad + \left\|\vartheta_\tau^{-1/2}\nabla\vartheta_\tau\right\|^2_{L^2(0,t_m;L^2(\Omega))} + \frac{1}{8}\left\|\nabla\left(\ln\vartheta_\tau - \ln\vartheta_{\mathcal{H},\tau}\right)\right\|^2_{L^2(0,t_m;L^2(\Omega))} \\
&\quad + \frac{1}{4}\left\|\partial_t\widehat{\chi}_\tau\right\|^2_{L^2(0,t_m;L^2(\Omega))} + \frac{1}{2}\left\|\nabla\chi^m\right\|^2_{L^2(\Omega)} + \frac{1}{2}\sum_{i=1}^m \left\|\nabla\left(\chi^i - \chi^{i-1}\right)\right\|^2_{L^2(\Omega)} \\
&\leq 2\sum_{i=0}^{m-1} \left\|\vartheta^i - \vartheta^i_{\mathcal{H}}\right\|^2_{L^2(\Omega)} + \frac{1}{4}\int_\Omega \left(\vartheta^0(\ln\vartheta^0 - 1) + 1\right) + \frac{1}{2}\left\|\nabla\chi^0\right\|^2_{L^2(\Omega)} + Ct_m \\
&\quad + \frac{1}{8}\sum_{i=1}^m \left\|\vartheta^i_{\mathcal{H}} - \vartheta^{i-1}_{\mathcal{H}}\right\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega (\vartheta^m - \vartheta^m_{\mathcal{H}})\ln\vartheta^m_{\mathcal{H}} + \frac{1}{2}\int_\Omega \left(\vartheta^0 - \vartheta^0_{\mathcal{H}}\right)\ln\vartheta^0_{\mathcal{H}} \\
&\quad - \frac{1}{2}\sum_{i=0}^{m-1}\int_\Omega \left(\vartheta^i - \vartheta^i_{\mathcal{H}}\right)\left(\ln\vartheta^{i+1}_{\mathcal{H}} - \ln\vartheta^i_{\mathcal{H}}\right) + \frac{1}{2}\sum_{i=1}^m \int_\Omega \left(\vartheta^i_{\mathcal{H}} - \vartheta^{i-1}_{\mathcal{H}}\right)\ln\vartheta^i_{\mathcal{H}}
\end{aligned}
$$

We observe the following estimates

$$
\begin{aligned}
\int_\Omega (\vartheta^m - \vartheta^m_{\mathcal{H}})\ln\vartheta^m_{\mathcal{H}} &\leq \frac{1}{8}\left\|\vartheta^m - \vartheta^m_{\mathcal{H}}\right\|^2_{L^2(\Omega)} + 2\left\|\ln\vartheta^m_{\mathcal{H}}\right\|^2_{L^2(\Omega)} \\
\int_\Omega \left(\vartheta^i - \vartheta^i_{\mathcal{H}}\right)\left(\ln\vartheta^{i+1}_{\mathcal{H}} - \ln\vartheta^i_{\mathcal{H}}\right) &\leq \frac{1}{2}\int_\Omega \left(\vartheta^i - \vartheta^i_{\mathcal{H}}\right)^2 + \frac{1}{2}\int_\Omega \left(\ln\vartheta^{i+1}_{\mathcal{H}} - \ln\vartheta^i_{\mathcal{H}}\right)^2 \\
&\leq \frac{1}{2}\left\|\vartheta^i - \vartheta^i_{\mathcal{H}}\right\|^2_{L^2(\Omega)} + \frac{1}{2\vartheta_*^2}\int_\Omega \left(\vartheta^i_{\mathcal{H}} - \vartheta^{i-1}_{\mathcal{H}}\right)^2
\end{aligned}
$$

where we used the boundedness of $\vartheta_{\mathcal{H},\tau}$ and thus the Lipschitz continuity of ln on $[\vartheta_*, \vartheta^*]$. For introducing τ_i we use the Young inequality

$$\int_\Omega \left(\vartheta^i_{\mathcal{H}} - \vartheta^{i-1}_{\mathcal{H}}\right) \ln \vartheta^i_{\mathcal{H}} \leq \frac{1}{\tau_i} \int_\Omega \left(\vartheta^i_{\mathcal{H}} - \vartheta^{i-1}_{\mathcal{H}}\right)^2 + \frac{\tau_i}{4} \int_\Omega \left(\ln \vartheta^i_{\mathcal{H}}\right)^2.$$

We collect all norms also depending on the initial values and denote them by C. In addition we have supposed a uniform partition $\mathcal{P}$ and thus yields $\sigma\tau \leq \tau_i \leq \tau$ with $\sigma \in (0,1]$ for all $1 \leq i \leq N$. Hence we obtain the estimate

$$\begin{aligned}
&\frac{\sigma\tau}{8} \sum_{i=1}^{m} \tau_i \left\| \frac{\vartheta^i - \vartheta^{i-1}}{\tau_i} \right\|^2_{L^2(\Omega)} + \frac{1}{8} \left\| \vartheta^m - \vartheta^m_{\mathcal{H}} \right\|^2_{L^2(\Omega)} + \frac{1}{4} \int_\Omega \left(\vartheta^m (\ln \vartheta^m - 1) + 1\right) \\
&\quad + \left\| (\vartheta_\tau)^{-\frac{1}{2}} \nabla \vartheta_\tau \right\|^2_{L^2(0,t_m;L^2(\Omega))} + \frac{1}{8} \left\| \nabla \left(\ln \vartheta_\tau - \ln \vartheta_{\mathcal{H},\tau}\right) \right\|^2_{L^2(0,t_m;L^2(\Omega))} \\
&\quad + \frac{1}{4} \left\| \partial_t \widehat{\chi}_\tau \right\|^2_{L^2(0,t_m;L^2(\Omega))} + \frac{1}{2} \left\| \nabla \chi^m \right\|^2_{L^2(\Omega)} + \frac{\sigma\tau}{2} \sum_{i=1}^{m} \tau_i \left\| \nabla \frac{\chi^i - \chi^{i-1}}{\tau_i} \right\|^2_{L^2(\Omega)} \\
&\leq 3 \sum_{i=0}^{m-1} \left\| \vartheta^i - \vartheta^i_{\mathcal{H}} \right\|^2_{L^2(\Omega)} + (\tau C + 1) \sum_{i=1}^{m} \tau_i \left\| \frac{\vartheta^i_{\mathcal{H}} - \vartheta^{i-1}_{\mathcal{H}}}{\tau_i} \right\|^2_{L^2(\Omega)} + \sum_{i=1}^{m} \tau_i \left\| \ln \vartheta_{\mathcal{H}} \right\|^2_{L^2(\Omega)}
\end{aligned}$$

The last step is to apply the discrete version of the Gronwall Lemma (B.2) to control the term $\|\vartheta^m - \vartheta^m_{\mathcal{H}}\|_{L^2(\Omega)}$. In addition we use the notation involving the linear interpolation for expressing the difference quotient pointed out in the Definition 3.10. Finally, using the Poincaré inequality, we have concluded the following norm estimates

$$\begin{aligned}
&\tau \left\| \partial \widehat{\vartheta}_\tau \right\|^2_{L^2(0,T;L^2(\Omega))} + \left\| \vartheta_\tau \right\|^2_{L^\infty(0,T;L^2(\Omega))} + \sup_{t \in [0,T]} \int_\Omega \left(\vartheta_\tau (\ln \vartheta_\tau - 1) + 1\right) \\
&\quad + \left\| \vartheta_\tau^{-1/2} \nabla \vartheta_\tau \right\|^2_{L^2(0,T;L^2(\Omega))} + \left\| \ln \vartheta_\tau \right\|^2_{L^2(0,T;H^1(\Omega))} \\
&\quad + \left\| \partial_t \widehat{\chi}_\tau \right\|^2_{L^2(0,T;L^2(\Omega))} + \left\| \chi_\tau \right\|^2_{L^\infty(0,T;H^1(\Omega))} + \tau \left\| \partial_t \widehat{\chi}_\tau \right\|^2_{L^2(0,T;H^1(\Omega))} \\
&\leq c + \left\| \partial \widehat{\vartheta}_{\mathcal{H},\tau} \right\|^2_{L^2(0,T;L^2(\Omega)} + \left\| \ln \vartheta_{\mathcal{H},\tau} \right\|^2_{L^2(0,T;L^2(\Omega)} \leq c,
\end{aligned} \tag{6.3}$$

due to the boundedness of $\vartheta_{\mathcal{H},\tau}$ in suitable norms given in Proposition 4.37.

Consequences

By comparing in (5.1a) we achieve

$$\widehat{\vartheta}_\tau \in H^1(0,T;H^{-1}(\Omega)) \quad \text{with} \quad \left\| \partial \widehat{\vartheta}_\tau \right\|_{L^2(0,T;H^{-1}(\Omega))} \leq c. \tag{6.4}$$

Also in comparison to (5.1b) we can even conclude

$$\chi_\tau \in L^2(0,T;H^2(\Omega)) \quad \text{with} \quad \|\chi_\tau\|_{L^2(0,T;H^2(\Omega))} \leq c. \tag{6.5}$$

Additionally we can prove in the same way already shown for (5.12)

$$\nabla \vartheta_\tau \in L^2(0,T;L^{4/3}(\Omega)) \quad \text{with} \quad \|\nabla \vartheta_\tau\|_{L^2(0,T;L^{4/3}(\Omega))} \leq c. \tag{6.6}$$

1.2. Second a priori estimate

At first we remind of the chain rule and the notation for β^i and its derivatives

$$\nabla\beta^i(\vartheta^i) = {\beta^i}'(\vartheta^i)\nabla\vartheta^i + \beta^i_{,x}(\vartheta^i).$$

We are testing the equation of ϑ^i (5.1a) by $\beta^i(\vartheta^i) - \beta^i(\vartheta^i_{\mathcal{H}})$. The term $\beta^i(\vartheta^i_{\mathcal{H}})$ allows us to integrate by parts without a remainder. It holds for the Laplacian an analogue formula like in the second a priori estimate for the limit in ε

$$\int_\Omega \Delta(\ln\vartheta^i)(\beta^i(\vartheta^i) - \beta^i(\vartheta^i_{\mathcal{H}})) = \int_\Omega \nabla(\ln\vartheta^i)\cdot\left[{\beta^i}'(\vartheta^i)\nabla\vartheta^i + \beta^i_{,x}(\vartheta^i) + \nabla\beta^i(\vartheta^i_{\mathcal{H}})\right].$$

By integrating and using the above stated expression we achieve

$$\begin{aligned}
&\frac{1}{\tau_i}\int_\Omega \left(\vartheta^i - \vartheta^{i-1}\right)\beta^i(\vartheta^i) + \int_\Omega \nabla\left(\ln\vartheta^i\right)\cdot{\beta^i}'(\vartheta^i)\nabla\vartheta^i + \int_\Omega \left(\beta^i(\vartheta^i) - \beta^i(\vartheta^i_{\mathcal{H}})\right)^2 \\
&= \frac{1}{\tau_i}\int_\Omega \left(\vartheta^i - \vartheta^{i-1}\right)\beta^i(\vartheta^i_{\mathcal{H}}) - \int_\Omega \nabla\ln\vartheta^i\cdot\beta^i_{,x}(\vartheta^i) + \int_\Omega \nabla\left(\ln\vartheta^i\right)\cdot\nabla\beta^i(\vartheta^i_{\mathcal{H}}) \\
&\quad + \int_\Omega \left(G'(\chi^i)\frac{\chi^i - \chi^{i-1}}{\tau_i} - \beta^i(\vartheta^i_{\mathcal{H}}) + \overline{\pi}^i(\vartheta^{i-1})\right)\left(\beta^i(\vartheta^i) - \beta^i(\vartheta^i_{\mathcal{H}})\right).
\end{aligned} \tag{6.7}$$

We define the primitive of β in the following way

$$B^i(r) := \int_1^r \beta^i(s)\,\mathrm{d}s \tag{6.8}$$

Then, observing by (3.2g) that $\beta(x,t,\cdot)$ is nondecreasing, it turns out that B^i is nonnegative, due to (3.2i), and convex with respect to r.

Therewith, we argue with the monotonicity of β^i for the first integral

$$\frac{1}{\tau_i}\int_\Omega \left(\vartheta^i - \vartheta^{i-1}\right)\beta^i(\vartheta^i) \geq \frac{1}{\tau_i}\int_\Omega\int_{\vartheta^{i-1}}^{\vartheta^i} \beta^i(s)\,\mathrm{d}s = \frac{1}{\tau_i}\int_\Omega \left(B^i(\vartheta^i) - B^{i-1}(\vartheta^{i-1})\right).$$

Now, we handle the right-hand side. With the help of discrete integration by parts is the first integral transformed to

$$\frac{1}{\tau_i}\int_\Omega \left(\vartheta^i - \vartheta^{i-1}\right)\beta^i(\vartheta^i_{\mathcal{H}}) = \frac{1}{\tau_i}\int_\Omega \left[\vartheta^i\beta^i(\vartheta^i_{\mathcal{H}}) - \vartheta^{i-1}\left(\beta^i(\vartheta^i_{\mathcal{H}}) - \beta^{i-1}(\vartheta^{i-1}_{\mathcal{H}})\right) - \vartheta^{i-1}\beta^{i-1}(\vartheta^{i-1}_{\mathcal{H}})\right].$$

The next two integrals can be estimated in the same way as already done in (5.13) the uniform estimate with respect to ε. So we first observe

$$\begin{aligned}
-\int_\Omega \nabla\ln\vartheta^i\cdot\beta^i_{,x}(\vartheta^i) &\leq 4\left\|\nabla\ln\vartheta^i\right\|^2_{L^2(\Omega)} + \frac{1}{16}\int_\Omega \left(\beta^i(\vartheta^i) + 1\right)^2 \\
&\leq 4\left\|\ln\vartheta^i\right\|^2_{H^1(\Omega)} + \frac{1}{8}\int_\Omega \left(\beta^i(\vartheta^i) - \beta^i(\vartheta^i_{\mathcal{H}})\right)^2 + \frac{1}{8}\left\|\beta^i(\vartheta^i_{\mathcal{H}}) + 1\right\|^2_{L^2(\Omega)}.
\end{aligned}$$

The next integral becomes by the Young Inequality B.3

$$\int_\Omega \nabla\left(\ln\vartheta^i\right)\cdot\nabla\beta^i(\vartheta^i_{\mathcal{H}}) \leq \frac{1}{2}\left\|\ln\vartheta^i\right\|^2_{H^1(\Omega)} + \frac{1}{2}\left\|\beta^i(\vartheta^i_{\mathcal{H}})\right\|^2_{H^1(\Omega)}.$$

Finally yields for the last integral

$$\int_\Omega \left(G'(\chi^i) \frac{\chi^i - \chi^{i-1}}{\tau_i} - \beta^i(\vartheta^i_{\mathcal{H}}) + \overline{\pi}^i(\vartheta^{i-1}) \right) \left(\beta^i(\vartheta^i) - \beta^i(\vartheta^i_{\mathcal{H}})\right)$$
$$\leq 4L_G^2 \left\| \frac{\chi^i - \chi^{i-1}}{\tau_i} \right\|^2_{L^2(\Omega)} + \left\|\beta^i(\vartheta^i_{\mathcal{H}})\right\|^2_{L^2(\Omega)} + \left\|\overline{\pi}^i(\vartheta^{i-1})\right\|^2_{L^2(\Omega)} + \frac{1}{8}\int_\Omega \left(\beta^i(\vartheta^i) - \beta^i(\vartheta^i_{\mathcal{H}})\right)^2,$$

where we actually remark $\left\|\overline{\pi}^i(\vartheta^{i-1})\right\|^2_{L^2(\Omega)} \leq L_\pi^2 \left\|\vartheta^{i-1}\right\|^2_{L^2(\Omega)} + \left\|\overline{\pi}^i_0\right\|^2_{L^2(\Omega)}$. Again by combining all these inequalities we observe

$$\frac{1}{\tau_i}\int_\Omega \left(B^i(\vartheta^i) - B^{i-1}(\vartheta^{i-1})\right) + \frac{1}{2}\left\|\beta^i(\vartheta^i) - \beta^i(\vartheta^i_{\mathcal{H}})\right\|^2_{L^2(\Omega)}$$
$$\leq \frac{1}{\tau_i}\int_\Omega \left[\vartheta^i\beta^i(\vartheta^i_{\mathcal{H}}) - \vartheta^{i-1}\left(\beta^i(\vartheta^i_{\mathcal{H}}) - \beta^{i-1}(\vartheta^{i-1}_{\mathcal{H}})\right) - \vartheta^{i-1}\beta^{i-1}(\vartheta^{i-1}_{\mathcal{H}})\right]$$
$$+ 5\left\|\ln\vartheta^i\right\|^2_{H^1(\Omega)} + \left\|\beta^i(\vartheta^i_{\mathcal{H}})\right\|^2_{L^2(\Omega)} + L_\pi^2\left\|\vartheta^{i-1}\right\|^2_{L^2(\Omega)} + 4L_G^2\left\|\frac{\chi^i - \chi^{i-1}}{\tau_i}\right\|^2_{L^2(\Omega)} + c,$$

where c depends on Ω and π_0. Now, multiplying by τ_i and summing from 1 to N let the following estimate hold

$$\int_\Omega B^N(\vartheta^N) + \frac{1}{2}\left\|\beta_\tau(\vartheta_\tau) - \beta_\tau(\vartheta_{\mathcal{H},\tau})\right\|^2_{L^2(0,T;L^2(\Omega))}$$
$$\leq \int_\Omega \vartheta^N\beta^N(\vartheta^N_{\mathcal{H}}) - \sum_{i=1}^{N-1}\tau_i\int_\Omega \vartheta^i\frac{\beta^{i+1}(\vartheta^{i+1}_{\mathcal{H}}) - \beta^i(\vartheta^i_{\mathcal{H}})}{\tau_i} + \int_\Omega \vartheta^0\beta^0(\vartheta^0_{\mathcal{H}})$$
$$+ \int_\Omega B^0(\vartheta^0) + 5\left\|\ln\vartheta_\tau\right\|^2_{L^2(0,T;H^1(\Omega))} + \left\|\beta_\tau(\vartheta_{\mathcal{H},\tau})\right\|^2_{L^2(0,T;L^2(\Omega))}$$
$$+ L_\pi^2\left\|\vartheta_\tau\right\|^2_{L^2(0,T;L^2(\Omega)} + 4L_G^2\left\|\partial\widehat{\chi}_\tau\right\|^2_{L^2(0,T;L^2(\Omega))} + c,$$

where C now also depends on the initial values and the endtime T. For the last estimates we observe the following estimates

$$\int_\Omega \vartheta^N\beta^N(\vartheta^N_{\mathcal{H}}) \leq \left\|\vartheta_\tau\right\|^2_{L^\infty(0,T;L^2(\Omega))} + \left\|\beta_\tau(\vartheta_{\mathcal{H},\tau})\right\|^2_{L^\infty(0,T;L^2(\Omega))}$$
$$\sum_{i=1}^{N-1}\tau_i\int_\Omega \vartheta^i\frac{\beta^{i+1}(\vartheta^{i+1}_{\mathcal{H}}) - \beta^i(\vartheta^i_{\mathcal{H}})}{\tau_i} \leq \left\|\vartheta_\tau\right\|^2_{L^2(0,T;L^2(\Omega))} + \left\|\partial_t\widehat{\beta}_\tau(\vartheta_{\mathcal{H},\tau})\right\|^2_{L^2(0,T;L^2(\Omega)}$$
$$\int_\Omega \vartheta^0\beta^0(\vartheta^0_{\mathcal{H}}) \leq \left\|\vartheta_0\right\|^2_{L^2(\Omega)} + \left\|\beta_\tau(\vartheta_{\mathcal{H},\tau})\right\|_{L^\infty(0,T;L^2(\Omega))}$$
$$\int_\Omega B^0(\vartheta^0) \leq \int_\Omega \beta_0(\vartheta^0) \leq \beta_0(\vartheta^*)\left|\Omega\right|$$

Finally, we can conclude thanks to (6.3) and again the Proposition 4.37

$$\left\|\beta_\tau(\vartheta_\tau)\right\|_{L^2(0,T;L^2(\Omega))} \leq c. \tag{6.9}$$

2. Passing to the limit in τ

Thus there exits the following limits such that (maybe taking subsequences)

$$\vartheta_\tau \overset{*}{\rightharpoonup} \vartheta \qquad \text{weakly star in } L^\infty(0,T;L^2(\Omega)) \tag{6.10a}$$
$$\nabla\vartheta_\tau \rightharpoonup \nabla\vartheta \qquad \text{weakly in } L^2(0,T;L^{4/3}(\Omega)) \tag{6.10b}$$
$$\widehat{\vartheta}_\tau \overset{*}{\rightharpoonup} \widehat{\vartheta} \qquad \text{weakly star in } L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)) \tag{6.10c}$$
$$\ln\vartheta_\tau \rightharpoonup \ell \qquad \text{weakly in } L^2(0,T;H^1(\Omega)) \tag{6.10d}$$
$$\chi_\tau \overset{*}{\rightharpoonup} \chi \qquad \text{weakly star in } L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)) \tag{6.10e}$$
$$\widehat{\chi}_\tau \overset{*}{\rightharpoonup} \widehat{\chi} \qquad \text{weakly star in } L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)) \tag{6.10f}$$
$$\beta_\tau(\vartheta_\tau) \rightharpoonup \xi \qquad \text{weakly in } L^2(0,T;L^2(\Omega)) \tag{6.10g}$$

as $\tau \to 0$. The notation $\tau \to 0$ means, that there exists a family of partitions $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ with $\tau^n \to 0$ as $n \to \infty$.

Let us first estimate the difference $\widehat{\vartheta}_\tau - \vartheta_\tau$

$$\begin{aligned}\left\|\widehat{\vartheta}_\tau - \vartheta_\tau\right\|^2_{L^2(0,T;L^2(\Omega))} &\leq \sum_{i=1}^N \int_{t_{i-1}}^{t^i} (t-t_i)^2 \left\|\frac{\vartheta^i - \vartheta^{i-1}}{\tau_i}\right\|^2_{L^2(\Omega)} \\ &\leq \sum_{i=1}^N \frac{\tau_i^3}{3}\left\|\frac{\vartheta^i - \vartheta^{i-1}}{\tau_i}\right\|^2_{L^2(\Omega)} \leq \tau^2 \left\|\partial\widehat{\vartheta}_\tau\right\|^2_{L^2(0,T;L^2(\Omega))}\end{aligned}$$

In the same way for the difference $\widehat{\chi}_\tau - \chi_\tau$

$$\begin{aligned}\left\|\widehat{\chi}_\tau - \chi_\tau\right\|^2_{L^2(0,T;H^1(\Omega))} &\leq \sum_{i=1}^N \int_{t_{i-1}}^{t^i} (t-t_i)^2 \left\|\frac{\chi^i - \chi^{i-1}}{\tau_i}\right\|^2_{H^1(\Omega)} \\ &\leq \sum_{i=1}^N \frac{\tau_i^3}{3}\left\|\frac{\chi^i - \chi^{i-1}}{\tau_i}\right\|^2_{H^1(\Omega)} \leq \tau^2 \left\|\partial\widehat{\chi}_\tau\right\|^2_{L^2(0,T;H^1(\Omega))}\end{aligned}$$

Thanks to (6.3) we have

$$\left\|\widehat{\vartheta}_\tau - \vartheta_\tau\right\|_{L^2(0,T;L^2(\Omega))} \leq C\sqrt{\tau} \tag{6.11a}$$
$$\left\|\widehat{\chi}_\tau - \chi_\tau\right\|_{L^2(0,T;H^1(\Omega))} \leq C\sqrt{\tau} \tag{6.11b}$$

As a consequence we can gather from (6.10c) and (6.10f)

$$\vartheta = \widehat{\vartheta} \quad \text{in } L^\infty(0,T;L^2(\Omega)) \quad \text{and} \quad \chi = \widehat{\chi} \quad \text{in } L^\infty(0,T;H^1(\Omega)). \tag{6.12}$$

Using the same arguments as in the limit in ε, particularly by the fact $(4/3)^* = 12/5 > 2$ and also by the Gagliardo-Nirenberg-Sobolev inequality (B.10) we observe, that $W^{1,4/3}(\Omega)$ is compactly embedded in $L^2(\Omega)$ and thus

$$\vartheta_\tau \to \vartheta \quad \text{strongly in } L^2(0,T;L^2(\Omega)). \tag{6.13}$$

But therewith we can conclude

$$\ln \vartheta_\tau \to \ell \quad \text{strongly in } L^2(0,T;L^2(\Omega)). \tag{6.14}$$

In consequence the last two convergences lead to

$$\limsup_{\tau\to 0} \vartheta_\tau \ln \vartheta_\tau = \vartheta \ell. \tag{6.15}$$

Now all assumptions of the Proposition 4.6 are shown and we get

$$l \in \ln \vartheta \quad \text{and therefore} \quad \vartheta > 0 \text{ and } \ell = \ln \vartheta. \tag{6.16}$$

In addition to this we can conclude $\xi = \beta(\vartheta)$ by Lemma 4.30.

By recalling the Definition 4.26 the difference between the translated and non translated step approximation can be estimated in the following way

$$\begin{aligned}\|\vartheta_\tau - \mathcal{T}_{-1}(\vartheta_\tau)\|^2_{L^2(Q)} &\leq \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \left\|\vartheta^i - \vartheta^{i-1}\right\|^2_{L^2(\Omega)} \leq \tau^2 \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \left\|\frac{\vartheta^i - \vartheta^{i-1}}{\tau^i}\right\|^2_{L^2(\Omega)} \\ &\leq \tau^2 \left\|\partial_t \widehat{\vartheta}_\tau\right\|^2_{L^2(Q)} \leq C\tau\end{aligned}$$

Finally using our norm estimate (6.3) and (6.13) we can conclude

$$\mathcal{T}_{-1}(\vartheta_\tau) \to \vartheta \quad \text{strongly in } L^2(0,T;L^2(\Omega)).$$

Now the assumptions of the Lemma 4.29 are fulfilled by $\overline{\pi}_\tau(\mathcal{T}_{-1}(\vartheta_\tau))$ and we obtain

$$\overline{\pi}_\tau(\vartheta_\tau) \to \pi(\vartheta) \quad \text{strongly in } L^2(0,T;L^2(\Omega)). \tag{6.17}$$

Finally, the remaining nonlinear terms (i.e. those related to G, F' and G') can be identified by using the convergences given in (6.12) accounting to our assumptions (3.2b).

Chapter 7

Boundedness and regularity

In this section we prove the Theorem 3.16 and the Proposition 3.17. These results were already obtained in [3]. The proof uses a *Moser type technique*. We will start with an a priori norm estimate for ϑ. From this point we continue searching iterative bounds in $L^p(Q)$ for ϑ. Thereby, we will obtain a sequence of Lebesgue exponents p_k with $p_k = \sigma^k$, where $\sigma > 1$. If we can show the boundedness of the series with all L^{p_k} norms of ϑ on Q, we will have proven the boundedness of ϑ.

At first we introduce some special notation.

Notation 7.1. If v is a real function or a real number we denote by

$$v^+ := \max\{0, v\}$$

its *positive part*. Further, n is a positive integer and δ a positive parameter, say $\delta \in (0, 1)$. Just another abbreviation is

$$u^* := \ln \vartheta^*.$$

Finally, we assume without loss of generality $q \leq 4$, in comparison to (3.19).

1. Prerequisites

1.1. Higher regularity

As already mentioned in Remark 3.4 the functions $G(\chi)$, $F'(\chi)$ and $G'(\chi)$ are smoother due to the boundedness and regularity of χ. In detail we have

$$G(\chi), F'(\chi), G'(\chi) \in L^\infty(Q) \cap L^2(0, T; V) \cap H^1(0, T; H).$$

Now, we obtain by comparison in the equation for ϑ (3.7f)

$$\partial_t \vartheta \in L^2(0, T; H^{-1}(\Omega)). \tag{7.1}$$

With this regularity ϑ fulfils the assumptions the of Lemma 4.25 and in the a priori estimates we can now use this weak chain rule.

1.2. Two more estimates

Lemma 7.2. *Set*

$$\phi_n(r) := \int_{\theta^*}^{r} \left(e^{2\min\left\{n,(\ln s-u^*)^+\right\}} - 1\right) \mathrm{d}s \quad \textit{for } r \in (0,+\infty). \tag{7.2a}$$

Then, positive constants α_* and C^* exist such that

$$\phi_n(r) \geq \alpha_* e^{3\min\left\{n,(\ln r-u^*)^+\right\}} - C^* \tag{7.2b}$$

for every $r \in (0,+\infty)$ and any positive integer n.

Proof. Assume at first $\vartheta^* \leq r \leq \vartheta^* e^n$. Then, we have

$$\phi_n(r) = \int_{\vartheta^*}^{r} \left(\left(\frac{s}{\vartheta^*}\right)^2 - 1\right) \mathrm{d}s = \frac{\vartheta^*}{3}\left(\frac{r}{\vartheta^*}\right)^3 - r + \frac{2\vartheta^*}{3} \geq \frac{\vartheta^*}{6}\left(\frac{r}{\vartheta^*}\right)^3 - C^*$$

for some $C^* > 0$, whence $\alpha_* := \vartheta^*/6$ works in (7.2b) for this case. Moreover, we can assume $C^* \geq \alpha_*$, so that (7.2b) holds even for $r \in (0, \vartheta^*)$, since $\phi_n(r) = 0$ for such values of r. Finally, if $r \geq \vartheta^* e^n$, we have $r \geq r' := \vartheta^* e^n$ and we already know that (7.2b) holds with $r = r'$. We deduce that

$$\phi_n(r) \geq \phi_n(r') \geq \alpha_* e^{3\min n,(\ln r'-u^*)^+} - C^* = \alpha_* e^{3n} - C^* = \alpha_* e^{3\min n,(\ln r-u^*)^+} - C^*.$$

□

Lemma 7.3. *Assume $p \in [1,+\infty)$ and set*

$$\psi_n(r) := \int_{\theta^*}^{r} \left(\min\left\{n,(\ln s-u^*)^+\right\}^{2p-1} - 1\right) \mathrm{d}s \quad \textit{for } r \in (0,+\infty). \tag{7.3a}$$

Then, ϕ_n satisfies

$$\psi(r) \geq \frac{1}{2p}\min\left\{n,(\ln r-u^*)^+\right\}^{2p} \quad \textit{for every } r \in (0,+\infty). \tag{7.3b}$$

Proof. If $\vartheta^* \leq r \leq \vartheta^* e^n$, we have

$$\psi_n(r) = \int_{u^*}^{\ln r} e^y (y-u^*)^{2p-1}\,\mathrm{d}y \geq \int_{u^*}^{\ln r} (y-u^*)^{2p-1}\,\mathrm{d}y = \frac{1}{2p}(\ln r - u^*)^{2p}$$

and (7.3b) follow. If instead $r \geq \vartheta^* e^n$ we observe that $\psi_n(r) \geq \psi_n(r')$, where $r' := \vartheta^* e^n$. On the other hand, we already know that (7.3b) holds with $r = r'$. Hence, we easily conclude that the desired inequality is true even in this case. Finally, if $r < \vartheta^*$, we have $\phi_n(r) = 0$ and (7.3b) trivially holds. □

2. First a priori estimate

We set

$$u := \ln \vartheta, \quad w_n := \min\left\{n, (u-u^*)^+\right\} \quad \text{and} \quad v_n := e^{2w_n} - 1. \tag{7.4a}$$

We want to use v_n as a test function in the equation for ϑ (3.7f). In fact, v_n is the truncation from below and above of ϑ^2. We check the assumptions for Lemma 4.25, where we set $\phi := \phi_n$ given by (7.2a). So we note that $u \in L^2(0,T;H^1(\Omega))$ and that $v_n = \phi_n(u)$, where ϕ_n is a Lipschitz continuous function. Hence, $v_n \in L^2(0,T;H^1(\Omega))$. Moreover, v_n vanishes on the boundary since $\ln\vartheta \le u^*$ by (3.6a). Therefore, even $v_n \in L^2(0,T;H^1_0(\Omega))$. Furthermore, ϕ_n is a C^1 convex function on $(0,+\infty)$ and $v_n = \phi_n'(\vartheta)$. Hence, we are allowed both: to test (3.7f) by v_n and to apply Lemma 4.25. We note that $\phi_n(\vartheta_0) = 0$ since $\vartheta_0 \le \vartheta^*$ in Ω by (3.6b). Then, we extend $\phi_n(r)$ with 0 value for $r = 0$ like done in (4.17) and set

$$f := -\beta(\vartheta^*) + \pi(\vartheta) + \partial G(\chi). \tag{7.4b}$$

Therewith we obtain from the chain rule in Lemma 4.25

$$\int_\Omega \phi_n(\vartheta(t)) + \int_{Q_t} \nabla u\cdot\nabla v_n + \int_{Q_t} \left(\beta(\vartheta) - \beta(\vartheta^*)\right) v_n = \int_{Q_t} f v_n. \tag{7.5}$$

We treat each integral separately. For the first one, we have

$$\int_\Omega \phi_n(\vartheta(t)) \ge \alpha_* \int_\Omega e^{3w_n(t)} - C^* = \alpha_* \left\| e^{w_n(t)}\right\|^3_{L^3(\Omega)} - c$$

thanks to Lemma 7.2. For the next term of (7.5) by observing that $\nabla u = \nabla w_n$ if and only if $\nabla v_n \ne 0$ we conclude

$$\int_{Q_t} \nabla u\cdot\nabla v_n = \int_{Q_t} \nabla w_n\cdot\nabla v_n = 2\int_{Q_t} e^{2w_n}\left|\nabla w_n\right|^2 = 2\int_{Q_t}\left|\nabla e^{w_n}\right|^2.$$

The last term on the left-hand side is also nonnegative. Indeed, v_n is nonnegative and $\beta(\vartheta) \ge \beta(\vartheta^*)$ where $v_n > 0$ since β is monotone. Thus, let us consider the right-hand side. We first notice that $f \in L^2(Q)$ due to our existence result. Now, by using the general Hölder Inequality B.5 with exponents 1/2, 1/3 and 1/6 as well as the Sobolev and Poincaré inequalities pointed out in Proposition B.9, we have

$$\begin{aligned}
\int_{Q_t} f v_n &\le \int_{Q_t} |f| \left(\left|e^{w_n}\right|^2 + 1\right) \\
&\le \|f\|_{L^1(Q_t)} + \int_0^t \|f(s)\|_{L^2(\Omega)} \left\|e^{w_n(s)}\right\|_{L^3(\Omega)} \left\|e^{w_n(s)}\right\|_{L^6(\Omega)} \,\mathrm{d}s \\
&\le c + \frac{1}{M_\Omega^2}\int_0^t \left\|e^{w_n(s)}\right\|^2_{L^6(\Omega)} \,\mathrm{d}s + c\int_0^t \|f(s)\|^2_{L^2(\Omega)} \left\|e^{w_n(s)}\right\|^2_{L^3(\Omega)} \,\mathrm{d}s \\
&\le \frac{1}{M_\Omega^2}\int_0^t \left\|e^{w_n(s)} - 1\right\|^2_{L^6(\Omega)} \,\mathrm{d}s + c\int_0^t \|f(s)\|^2_{L^2(\Omega)} \left\|e^{w_n(s)}\right\|^2_{L^3(\Omega)} \,\mathrm{d}s + c \\
&\le \int_{Q_t} \left|\nabla e^{w_n}\right|^2 + c\int_0^t \|f(s)\|^2_{L^2(\Omega)} \left\|e^{w_n(s)}\right\|^3_{L^3(\Omega)} \,\mathrm{d}s + c.
\end{aligned}$$

Note that we have changed the exponent of the norm $\left\|e^{w_n(s)}\right\|_{L^3(\Omega)}$ in the last two estimates from 2 to 3 due to $e^{w_n} \geq 1$ a.e. in Q_t. At this point we collect all the estimates above and observe

$$\alpha_* \left\|e^{w_n(t)}\right\|_{L^3(\Omega)}^3 + \int_{Q_t} |\nabla e^{w_n}|^2 \leq c \int_0^t \|f(s)\|_{L^2(\Omega)}^2 \left\|e^{w_n(s)}\right\|_{L^3(\Omega)}^3 \mathrm{d}s + c.$$

Noting that $\|f(\cdot)\|_{L^2(\Omega)}^2 \in L^1(0,T)$ since $f \in L^2(\Omega)$ we can apply the Gronwall Lemma B.1 and obtain

$$\|e^{w_n}\|_{L^\infty(0,T;L^3(\Omega))}^3 + \int_Q |\nabla e^{w_n}|^2 \leq c. \tag{7.6}$$

Consequence. From (7.6) and (B.12a) we deduce that $\exp(w_n)$ is bounded in $L^4(Q)$. Hence, we can let n tend to infinity and infer that $\exp\left((u-u^*)^+\right) \in L^4(Q)$. Now, we observe that

$$e^{(u-u^*)^+} = e^{-u^*} e^u = \frac{\vartheta}{\vartheta^*} \quad \text{where} \quad \vartheta > \vartheta^*$$

In addition, because of ϑ is positive, we conclude that

$$\vartheta \in L^4(Q). \tag{7.7}$$

Now, we rewrite the equation for χ (3.7g) at the form

$$\partial_t \chi - \Delta \chi = -F'(\chi) - G'(\chi)\vartheta$$

and observe that the right-hand side belongs to $L^4(Q)$ due to (7.6) and the Lipschitz continuity of F and G. By the general theory for parabolic equations [18], we can conclude $\partial_t \chi \in L^4(Q)$ and therewith also

$$\|\partial_t G(\chi)\|_{L^q(Q)} \leq c \|\partial_t G(\chi)\|_{L^4(Q)} < +\infty \tag{7.8}$$

due to our assumption $q < 4$ in the beginning of this chapter. On the other hand, we can estimate the right-hand side of the equation for ϑ (3.7f) in a better way. Indeed, recalling the assumptions on π (3.2j) and (3.2k) we conclude that

$$\|\pi(\vartheta)\|_{L^q(Q)} \leq c \|\vartheta\|_{L^4(Q)} + \|\pi_0\|_{L^q(Q)} < +\infty. \tag{7.9}$$

3. The Moser type procedure

Starting from the just obtained estimates we want to increase the exponent $p \in [1, +\infty)$ for the L^p estimate on Q in an iterative scheme for

$$w := (\ln \vartheta - u^*)^+. \tag{7.10}$$

We declare that the values of the constant c do not depend on p. We define

$$u := \ln \vartheta, \quad w_n := \min\left\{(u-u^*)^+, n\right\}, \quad \text{and} \quad v_n := w_n^{2p-1}. \tag{7.11}$$

The estimate is similar to the first a priori one. We observe that $v_n \in L^2(0,T;H_0^1(\Omega))$ due to the assumptions on the boundary values (3.6a). Thus, v_n is an admissible test function for (3.7f) and lemma (4.25) can be once more applied with $\Phi = \psi_n$ given by (7.3a). We note that also holds $v_n = \psi'(\vartheta)$. As $\psi_n(\vartheta_0) = 0$ due to the assumption on the initial data (3.6b), we obtain

$$\int_\Omega \psi_n(\vartheta(t)) + \int_{Q_t} \nabla u \cdot \nabla v_n + \int_{Q_t} (\beta(\vartheta) - \beta(\vartheta^*))\, v_n = \int_{Q_t} f v_n, \tag{7.12}$$

where f is given by (7.4b). Thanks to Lemma 7.3, we immediately derive that

$$\int_\Omega \psi_n(\vartheta(t)) \geq \frac{1}{2p} \int_\Omega (w_n(t))^{2p}.$$

The next term on the left-hand side can be easily treated with the observation $\nabla u = \nabla w_n$ if and only if $\nabla v_n \neq 0$

$$\int_{Q_t} \nabla u \cdot \nabla v = (2p-1) \int_{Q_t} w_n^{2p-2} |\nabla w_n|^2 = \frac{2p-1}{p^2} \int_{Q_t} |\nabla w_n^p|^2 \geq \frac{1}{p} \int_{Q_t} |\nabla w_n^p|^2 .$$

In addition the last integral is nonnegative since $v_n \geq 0$ and $\beta(\vartheta^*) \geq \beta(\vartheta)$ only if $v_n = 0$. In order to deal with the right-hand side, let us observe that f belongs to $L^q(Q)$ thanks to (7.8) and (7.9) as well as the assumptions (3.3) on β. By denoting with q' the conjugate exponent of q, such that $1/q + 1/q' = 1$ and the applying the Hölder inequality, we obtain

$$\int_{Q_t} f v_n \leq \|f\|_{L^q(Q)} \|v_n\|_{L^{q'}(Q)} \leq c \left\| w_n^{2p-1} \right\|_{L^{q'}(Q)} = c \left\| w_n^p \right\|_{L^{q'(2p-1)/p}(Q)}^{(2p-1)/p} .$$

Collecting the above estimates, we obtain

$$\|(w_n(t))^p\|_{L^2(\Omega)}^2 + \int_{Q_t} |\nabla w_n^p|^2 \leq cp \left\| w_n^p \right\|_{L^{q'(2p-1)/p}(Q)}^{(2p-1)/p} \quad \text{for every } t \in [0,T].$$

By using the parabolic embedding pointed out in (B.12b), we obtain the following estimate

$$c \left\| w_n^p \right\|_{L^{10/3}(\Omega)}^2 \leq \|w_n^p\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla w_n^p\|_{L^2(Q)}^2 \leq cp \left\| w_n^p \right\|_{L^{q'(2p-1)/p}(Q)}^{(2p-1)/p} .$$

At this point, we note that (7.10) and $\vartheta \in L^2(Q)$ imply that $w \in L^r(Q)$ for every $r \in [1, +\infty)$ due to the logarithm. So we can let n tend to infinity and conclude that

$$\|w\|_{L^{10p/3}(\Omega)} \leq (cp)^{1/(2p)} \|w\|_{L^{q'(2p-1)}(Q)}^{(2p-1)/(2p)} ,$$

where we also have taken the square root and point out that w is given by (7.10). Applying the Hölder inequality with the exponents

$$\frac{1}{2pq'} + \frac{1}{2p(2p-1)q'} = \frac{1}{(2p-1)q'}$$

to the right-hand side results in

$$\|w\|_{L^{q'(2p-1)}(Q)}^{(2p-1)/(2p)} \leq |Q|^{1/(4p^2q')} \|w\|_{L^{2pq'}(Q)}^{(2p-1)/(2p)} = \left(|Q|^{1/(2pq')}\right)^{1/2p} \|w\|_{L^{2pq'}(Q)}^{(2p-1)/(2p)} ,$$

where $|Q|^{1/(2pq')}$ is bounded with respect to p. We get by combining the last two inequalities

$$\|w\|_{L^{10p/3}(\Omega)} \leq (cp)^{1/(2p)} \|w\|_{L^{2pq'}(Q)}^{(2p-1)/(2p)} , \tag{7.13}$$

where we, without loss of generality, assume $c \geq 1$.

4. Conclusion

We rewrite (7.13) in the form of

$$\|w\|_{L^{\sigma 2pq'}(\Omega)} \leq (cp)^{1/(2p)} \|w\|_{L^{2pq'}(Q)}^{(2p-1)/(2p)} \quad \text{where} \quad \sigma := \frac{5}{3q'} \tag{7.14}$$

and observe that $\sigma > 1$ since $1/q' = 1 - 1/q < 3/5$ due to we assume $q > 5/2$ in (3.19). Now, we apply (7.14) to the divergent sequence $\{p_k\}$ defined by $p_k := \sigma^k$ and obtain the iterative estimate

$$\|w\|_{L^{2p_{k+1}q'}(Q)} \leq (cp_k)^{1/(2p_k)} \|w\|_{L^{2p_k q'}(Q)}^{(2p_k-1)/(2p_k)}. \tag{7.15}$$

Setting for convenience

$$\ell_k := \ln^+ \|w\|_{L^{2p_k q'}(Q)}$$

and taking the positive part of the logarithm on both sides of (7.15), we derive that

$$\ell_{k+1} \leq \frac{1}{2p_k} \ln(cp_k) + \frac{2p_k - 1}{2p_k} \ell_k \leq \frac{1}{2p_k} \ln(cp_k) + \ell_k,$$

where we want to remember, that we chose $c \geq 1$ and so the logarithm is nonnegative. Therewith, we find for every ℓ_k the following bound

$$\ell_k \leq \ell_0 + \sum_{i=1}^{\infty} \frac{1}{2p_i} \ln(cp_i) = \ell_0 + c \sum_{i=1}^{\infty} \frac{i}{\sigma^i} = \ell_0 + c\sigma \frac{\mathrm{d}}{\mathrm{d}\sigma} \sum_{i=1}^{\infty} -\frac{1}{\sigma^i} = \ell_0 + c \frac{\sigma}{(1-\sigma)^2} =: C.$$

Hence, we have found out a bound which is independent of k

$$\|w\|_{L^{2p_k q'}} (Q) \leq e^C. \tag{7.16}$$

Letting k tending to $+\infty$ and reminding of the definition of w in (7.10), we deduce

$$\vartheta \in L^\infty(Q). \tag{7.17}$$

5. Improved regularity

By using the two regularity conditions $\ln \vartheta \in L^2(0, T; H^1(\Omega))$ and the just proved boundedness, we see that $\nabla \vartheta^m = m\vartheta^m \nabla \ln \vartheta$ for every $m > 0$, whence

$$\vartheta^m \in L^2(0, T; H^1(\Omega)) \quad \text{for every } m \in (0, +\infty). \tag{7.18}$$

In addition, as $F'(\chi)$ and $G'(\chi)$ are bounded, we deduce from the regularity conditions of χ (3.7b) as well as its equation (3.7g) that $\partial_t \chi - \Delta \chi$ is bounded, too. Hence, we have

$$\chi \in L^p(0, T; W^{2,p}(\Omega)) \cap W^{1,p}(0, T; L^p(\Omega)) \quad \text{for every } p \geq 1 \tag{7.19}$$

thanks to the general theory of linear parabolic equations [18].

Chapter 8

Continuous dependence on the data

In this section, we prove the Theorem 3.18. At first, we point out some additional notation and tools.

1. Prerequisites

We want to remind of some facts and assumptions already stated in the chapter with the main results. In addition, we need again two tools. We will introduce the *Riesz operator*, which will be the most important tool in the chapter to obtain an estimate. It is necessary to have a special harmonic extension to handle the non homogeneous boundary values.

1.1. Assumptions and Notation

We have reinforced the conditions of the right hand side $R(x,t,r) := \pi(x,t,r) - \beta(x,t,r)$. We assumed that R is Lipschitz continuous with respect to the third variable, that is

$$|R(x,t,r_1) - R(x,t,r_2)| \le L_R \, |r_1 - r_2| \qquad \forall r_1, r_2 \in \mathbb{R}, \tag{8.1}$$

where L_R is its Lipschitz constant.

According to the Theorem 3.18 we take $\delta > 0$ and two solutions $(\vartheta_1, \chi_1, \xi_1)$ and $(\vartheta_2, \chi_2, \xi_2)$ such that the initial data $\vartheta_{0,1}, \vartheta_{0,2}, \chi_{0,1}$ and $\chi_{0,2}$ as well as the boundary data $\vartheta_{\Gamma,1}$ and $\vartheta_{\Gamma,2}$ satisfy

$$\|\vartheta_{\Gamma,1} - \vartheta_{\Gamma,2}\|_{L^2(0,T;L^2(\Gamma))} \le \delta \tag{8.2a}$$

$$\|\vartheta_{0,1} - \vartheta_{0,2}\|_{H^{-1}(\Omega)} \le \delta \tag{8.2b}$$

$$\|\chi_{0,1} - \chi_{0,2}\|_{H^1(\Omega)} \le \delta \tag{8.2c}$$

We can apply the Theorem 3.16 due to the assumptions on the two solutions $(\vartheta_1, \chi_1, \xi_1)$, $(\vartheta_2, \chi_2, \xi_2)$ and deduce that they are bounded by a constant M.

For convenience we wet $\vartheta := \vartheta_1 - \vartheta_2$ and $\chi := \chi_1 - \chi_2$.

1.2. Riesz operator

The main tool we use, is the operator $\mathbf{R} : H^{-1}(\Omega) \to H_0^1(\Omega)$ given by the *Riesz representation theorem*, see for example [10], namely

$$\text{for } v^* \in H^{-1}(\Omega) \text{ and } v \in H_0^1(\Omega), \quad v = \mathbf{R}v^* \quad \text{means} \quad v^* = -\Delta v. \tag{8.3}$$

We note that

$$\langle -\Delta u, \mathbf{R}v \rangle = \int_\Omega uv \quad \text{for every } u \in H_0^1(\Omega) \text{ and } v \in L^2(\Omega) \tag{8.4a}$$

$$\langle u^*, \mathbf{R}v^* \rangle = (u^*, v^*)_* \quad \text{for every } u^*, v^* \in H^{-1}(\Omega) \tag{8.4b}$$

$$\int_0^t \langle \partial_t u(s), \mathbf{R}u(s) \rangle \, \mathrm{d}s = \frac{1}{2} \|u(t)\|_*^2 - \frac{1}{2} \|u(0)\|_*^2$$
$$\text{for every } u \in H^1(0,T; H^{-1}(\Omega)) \text{ and for a.a. } t \in (0,T). \tag{8.4c}$$

Thereby, we denote with $\langle \cdot, \cdot \rangle : H^{-1}(\Omega) \times H_0^1(\Omega)$ the dual pairing between these spaces. In addition the norm $\|\cdot\|_*$ in formula (8.4c) denotes the norm in $H^{-1}(\Omega)$ dual to the norm $v \mapsto \|\nabla v\|_{L^2(\Omega)}$ in $H_0^1(\Omega)$ and the symbol $(\cdot,\cdot)_* : H^{-1}(\Omega) \times H^{-1}(\Omega) \to \mathbb{R}$ is the corresponding inner product. By the Poincaré inequality is this norm and product in $H^{-1}(\Omega)$ equivalent to the standard ones and we mainly use them for convenience. In particular, we denote by M_Ω a constant that makes the following relations true

$$\begin{aligned} \|v\|_{H^{-1}(\Omega)} &\le M_\Omega \|v\|_* \quad \text{for every } v \in H^{-1}(\Omega) \\ \text{and} \quad \|v\|_* &\le M_\Omega \|v\|_{L^2(\Omega)} \quad \text{for every } v \in L^2(\Omega) \end{aligned} \tag{8.5}$$

1.3. Harmonic extension

We want to define $l_\mathcal{H}$, the harmonic extension of $\ln \vartheta_{1,\Gamma} - \ln \vartheta_{2,\Gamma}$, such that

$$\Delta l_\mathcal{H} = 0 \quad \text{in } \Omega \quad \text{and} \quad l_\mathcal{H} = \ln \vartheta_{1,\Gamma} - \ln \vartheta_{2,\Gamma} \quad \text{on } \Gamma \tag{8.6}$$

We obtain by the general theory of elliptic partial differential equations and the compact embedding of $H^{-1/2}(\Gamma)$ in $L^2(\Gamma)$ the following estimate

$$\begin{aligned} \|l_\mathcal{H}\|_{L^2(0,T;L^2(\Omega))} &\le c \|\ln \vartheta_{1,\Gamma} - \ln \vartheta_{2,\Gamma}\|_{L^2(0,T;H^{-1/2}(\Gamma)} \\ &\le c \|\ln \vartheta_{\Gamma,1} - \ln \vartheta_{\Gamma,2}\|_{L^2(0,T;L^2(\Gamma))} \\ &\le \frac{c}{\vartheta_*} \|\vartheta_{\Gamma,1} - \vartheta_{\Gamma,2}\|_{L^2(0,T;L^2(\Gamma))} \le c\delta \end{aligned} \tag{8.7}$$

also thanks to the Lipschitz continuity of ln on $[\vartheta_*, \vartheta^*]$ and by the boundedness of $\vartheta_{i,\Gamma}$ on that interval for $i = 1, 2$.

2. A priori estimate

2.1. Preliminary considerations

We test the difference of (3.7f) by $\mathbf{R}\vartheta$ and the difference of (3.7g) by $\mu\partial_t\chi$. We add these equations and integrate over Q_t and use some properties of the Riesz operator. Therefore, we want to explain some of the occurring terms more in detail. The first term of (3.7f) becomes with the help of (8.4c)

$$\int_0^t \langle \partial_t\vartheta(s), \mathbf{R}\vartheta(s)\rangle \,\mathrm{d}s = \frac{1}{2}\left\|\vartheta(t)\right\|_*^2 - \frac{1}{2}\left\|\vartheta_{0,1} - \vartheta_{0,2}\right\|_*^2.$$

For the second term in (3.7f) we want to use the property (8.4a) of the Riesz operator. The problem is, that we do not know, whether $\ln\vartheta_1(t) - \ln\vartheta_2(t)$ belongs to $H_0^1(\Omega)$ for a.e. $t \in (0,T)$ or not. Therefore, we take the harmonic extension $l_{\mathcal{H}}$ introduced in (8.6), because

$$\ln\vartheta_1(t) - \ln\vartheta_2(t) - l_{\mathcal{H}}(t) \in H_0^1(\Omega) \quad \text{for a.e. } t \in (0,T)$$

and obtain by applying (8.4a)

$$\int_0^t \langle -\Delta\left(\ln\vartheta_1 - \ln\vartheta_2 - l_{\mathcal{H}}\right), \mathbf{R}\vartheta\rangle = \int_{Q_t} \left(\ln\vartheta_1 - \ln\vartheta_2 - l_{\mathcal{H}}\right)\vartheta,$$

where we also want to point out that $-\Delta l_{\mathcal{H}} = 0$ by the Definition (8.6)

2.2. Estimate

With these preliminary considerations it follows that

$$\begin{aligned}
&\frac{1}{2}\left\|\vartheta(t)\right\|_*^2 + \int_{Q_t}(\ln\vartheta_1 - \ln\vartheta_2)\vartheta + \mu\int_{Q_t}|\partial_t\chi|^2 + \frac{\mu}{2}\int_\Omega|\nabla\chi|^2 + \frac{\mu}{2}\int_\Omega|\chi(t)|^2 \\
&\leq \int_0^t(\partial_t G(\chi_1(s)) - \partial_t G(\chi_2(s)), \vartheta(s))_*\,\mathrm{d}s + \int_0^t((R(\vartheta_1) - R(\vartheta_2))(s), \vartheta(s))_*\,\mathrm{d}s \\
&\quad + \mu\int_{Q_t}(F'(\chi_2) - F'(\chi_1))\partial_t\chi + \mu\int_{Q_t}(G'(\chi_2)\vartheta_2 - G'(\chi_1)\vartheta_1)\partial_t\chi \\
&\quad + \frac{1}{2}\left\|\vartheta_{0,1} - \vartheta_{0,2}\right\|_*^2 + \frac{\mu}{2}\int_\Omega|\nabla(\chi_{0,1} - \chi_{0,2})| + \int_{Q_t} l_{\mathcal{H}}\vartheta + \frac{\mu}{2}\int_\Omega|\chi(t)|^2,
\end{aligned}$$

where we added the integral $\frac{\mu}{2}\int_\Omega|\chi(t)|^2$ to both sides. The only term on the left-hand side, which needs some treatment is the second one. We observe by the mean value theorem for derivatives

$$\frac{\ln\vartheta_1 - \ln\vartheta_2}{\vartheta_1 - \vartheta_2} = (\ln\xi)' = \frac{1}{\xi} \geq \frac{1}{M},$$

with $\xi \in (\min\{\vartheta_1, \vartheta_2\}, \max\{\vartheta_1, \vartheta_2\})$. In addition the term $\frac{1}{M}$ is a lower bound due to the boundedness $0 < \vartheta_i \leq M$ for $i = 1, 2$. Therewith we find

$$\int_{Q_t}(\ln\vartheta_1 - \ln\vartheta_2)\,\vartheta = \int_{Q_t}\frac{\ln\vartheta_1 - \ln\vartheta_2}{\vartheta_1 - \vartheta_2}|\vartheta|^2 \geq \frac{1}{M}\int_{Q_t}|\vartheta|^2$$

Now, we are dealing with the first term on the right-hand side

$$
\begin{aligned}
&\int_0^t (\partial_t G(\chi_1(s)) - \partial_t G(\chi_2(s)), \vartheta(s))_* \,\mathrm{d}s \\
&\quad = \int_0^t \big(G'(\chi_1(s))\partial_t\chi(s) + (G'(\chi_1(s)) - G'(\chi_2(s)))\partial_t\chi_2(s), \vartheta(s)\big)_* \\
&\quad \le M_\Omega \int_0^t \left\|G'(\chi_1(s))\partial_t\chi(s)\right\|_{L^2(\Omega)} \|\vartheta(s)\|_* \,\mathrm{d}s \\
&\qquad + M_\Omega \int_0^t \left\|G'(\chi_1(s)) - G'(\chi_2(s))\right\|_{L^4(\Omega)} \|\partial_t\chi_2(s)\|_{L^4(\Omega)} \|\vartheta(s)\|_* \,\mathrm{d}s \\
&\quad \le M_\Omega L \int_0^t \|\partial_t\chi(s)\|_{L^2(\Omega)} \|\vartheta(s)\|_* \,\mathrm{d}s + M_\Omega L \int_0^t \|\chi(s)\|_{L^4(\Omega)} \|\partial_t\chi_2(s)\|_{L^4(\Omega)} \|\vartheta(s)\|_* \,\mathrm{d}s \\
&\quad \le \frac{\mu}{8}\int_{Q_t} |\partial_t\chi|^2 + \frac{2M_\Omega^2 L^2}{\mu}\int_0^t \|\vartheta(s)\|_*^2 \,\mathrm{d}s \\
&\qquad + \int_0^t \|\chi(s)\|_{L^4(\Omega)}^2 + \frac{M_\Omega^2 L^2}{4}\int_0^t \|\partial_t\chi_2(s)\|_{L^4(\Omega)}^2 \|\vartheta(s)\|_*^2 \,\mathrm{d}s \\
&\quad \le \frac{\mu}{8}\int_{Q_t} |\partial_t\chi|^2 + \frac{2M_\Omega^2 L^2}{\mu}\int_0^t \|\vartheta(s)\|_*^2 \,\mathrm{d}s \\
&\qquad + M_\Omega^2 \int_{Q_t} |\nabla\chi|^2 + M_\Omega^2 \int_{Q_t} |\chi|^2 + \frac{M_\Omega^2 L^2}{4}\int_0^t \|\partial_t\chi_2(s)\|_{L^4(\Omega)}^2 \|\vartheta(s)\|_*^2 \,\mathrm{d}s,
\end{aligned}
$$

where we used in the last estimate the embedding Lemma B.13 by Lions already used in (5.4a). The next terms we can handle easier. Just with the assumptions on R holds

$$
\begin{aligned}
\int_0^t ((R(\vartheta_1) - R(\vartheta_2)(s), \vartheta(s))_* \,\mathrm{d}s &\le M_\Omega L_R \int_0^t \|\vartheta(s)\|_{L^2(\Omega)} \|\vartheta(s)\|_* \,\mathrm{d}s \\
&\le \frac{1}{2M}\int_{Q_t} |\vartheta|^2 + M M_\Omega^2 L_R^2 \int_0^t \|\vartheta(s)\|_*^2 \,\mathrm{d}s.
\end{aligned}
$$

Furthermore, we have by the Lipschitz continuity of F

$$
\mu \int_{Q_t} \big(F'(\chi_2) - F'(\chi_1)\big)\,\partial_t\chi \le \frac{\mu}{8}\int_{Q_t} |\partial_t\chi|^2 + 2\mu L_F^2 \int_{Q_t} \left|\chi^2\right|.
$$

We achieve with the Lipschitz continuity of G as well as the boundedness of ϑ_i for $i = 1, 2$

$$
\begin{aligned}
\mu \int_{Q_t} \big(G'(\chi_2)\vartheta_2 - G'(\chi_1)\vartheta_1\big)\,\partial_t\chi &= \mu \int_{Q_t} \big(G'(\chi_2) - G'(\chi_1)\big)\,\vartheta_1\partial_t\chi - \mu \int_{Q_t} G'(\chi_2)\vartheta\partial_t\chi \\
&\le \mu L M \int_{Q_t} |\chi|\,|\partial_t\chi| + \mu L \int_{Q_t} |\vartheta|\,|\partial_t\chi| \\
&\le \frac{\mu}{8}\int_{Q_t} |\partial_t\chi|^2 + 4\mu L^2 M^2 \int_{Q_t} \left|\chi^2\right| + 2\mu L \int_{Q_t} |\vartheta|^2.
\end{aligned}
$$

The integral we added to both sides will be rewritten and estimated in the following way

$$
\frac{\mu}{2}\int_\Omega |\chi(t)|^2 = \mu \int_{Q_t} \chi\partial_t\chi + \frac{\mu}{2}\int_\Omega |\chi_{0,1} - \chi_{0,2}|^2 \le \frac{\mu}{8}\int_{Q_t} |\partial_t\chi|^2 + \mu \int_{Q_t} |\chi|^2 + \frac{\mu\delta^2}{2},
$$

by the assumptions on the initial values of χ_1 and χ_2. The same arguments are holding for

$$\frac{1}{2}\|\vartheta_{0,1}-\vartheta_{0,2}\|_*^2 \le \frac{M_\Omega^2\delta^2}{2} \quad \text{and} \quad \frac{\mu}{2}\int_\Omega |\nabla(\chi_{0,1}-\chi_{0,2})|^2 \le \frac{\mu\delta^2}{2}.$$

Finally, we can estimate the last term with the harmonic extension

$$\int_{Q_t} l_{\mathcal{H}}\vartheta \le \frac{1}{4M}\int_{Q_t}|\vartheta|^2 + M\int_{Q_t}|\vartheta_{\mathcal{H}}|^2 \le \frac{1}{4M}\int_{Q_t}|\vartheta|^2 + c\delta^2.$$

2.3. Conclusion

Combining all these estimates and choosing $\mu = 1/(8ML^2)$ leads to

$$\begin{aligned}&\frac{1}{2}\|\vartheta(t)\|_*^2 + \frac{1}{2M}\|\vartheta\|_{L^2(Q_t)}^2 + \frac{1}{8}\|\partial_t\chi\|_{L^2(Q_t)} + \frac{\mu}{2}\|\nabla\chi(t)\|_{L^2(\Omega)}^2 + \frac{\mu}{2}\|\chi(t)\|_{L^2(\Omega)}^2\\ &\le c\int_0^t \left(1+\|\partial_t\chi_2(s)\|_{L^4(\Omega)}^2\right)\|\vartheta(s)\|_*^2 \,\mathrm{d}s + c\int_0^t \|\nabla\chi(s)\|_{L^2(\Omega)}^2 \,\mathrm{d}s + c\int_0^t \|\chi(s)\|_{L^2(\Omega)}^2 \,\mathrm{d}s + c\delta^2\end{aligned}$$

Due to the improved regularity obtained by the boundedness of ϑ in Theorem 3.17, we have $\|\partial_t\chi_2(\cdot)\|_{L^4(Q)}^2 \in L^1(0,T)$. Therefore, we can apply the Gronwall Lemma B.1 and set $t=T$. We can conclude with the following norm estimate

$$\|\vartheta\|_{L^\infty(0,T;H_0^1(\Omega)^*)\cap L^2(Q)} + \|\chi\|_{H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;H^1(\Omega))} \le c\delta, \tag{8.8}$$

which gives the continuous dependence on the data and the Theorem 3.18 is proved.

Appendix A

Measure theory

Definition A.1 (Carathéodory function)**.** A function $\pi : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function provided

$$\text{i)} \quad \text{the map } y \mapsto \pi(x,y) \text{ is continuous a.e. } x \in \Omega, \tag{A.1a}$$

$$\text{ii)} \quad \text{the map } x \mapsto \pi(x,y) \text{ is measurable for all } y \in \mathbb{R}. \tag{A.1b}$$

We denote by $|\cdot|$ the Lebesgue measure on $\mathbb{R}^n$.

Lemma A.2 (Jensen Inequality)**.** *Let Ω denote a domain in $\mathbb{R}^n$. Take a measurable convex function ϕ defined on the real axis and $f : \Omega \to \mathbb{R}$ an integrable function, then*

$$\phi\left(\frac{1}{|\Omega|}\int_\Omega f \,\mathrm{d}x\right) \le \frac{1}{|\Omega|}\int_\Omega \phi \circ f \,\mathrm{d}x. \tag{A.2}$$

Theorem A.3 (Severini-Egorov)**.** *Suppose $f_k : \mathbb{R}^n \to \mathbb{R}^m$ are integrable. Assume also $A \subset \mathbb{R}^n$ with $|A| < \infty$ and $f_k \to g$ a.e. on A. Then for each $\varepsilon > 0$ exists a set $B \subset A$ such that*

$$\textit{i)} \quad |A \backslash B| < \varepsilon, \ \textit{and} \tag{A.3a}$$

$$\textit{ii)} \quad f_k \to g \ \textit{uniformly on } B. \tag{A.3b}$$

Proof. See [11, Theorem 3, p. 16]. □

Theorem A.4 (Lebesgue-Besicovitch Differentiation Theorem)**.** *Take $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then*

$$\lim_{r\to 0} \frac{1}{|U(x,r)|}\int_{U(x,r)} f \,\mathrm{d}y = f(x) \quad \textit{for a.e. } x \in \mathbb{R}^n, \tag{A.4a}$$

where $U(x,r)$ denotes the ball around x with radius r.

Especially if $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ for some $1 \le p < \infty$ yields

$$\lim_{U \downarrow \{x\}} \frac{1}{|U|}\int_U |f - f(x)|^p \,\mathrm{d}y = 0 \quad \textit{for a.e. } x \in \mathbb{R}^n. \tag{A.4b}$$

Proof. See [11, p. 43-44]. □

Lemma A.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain. For $1 < q < \infty$ denote g a function in $L^q(\Omega)$ as well as $\{g_k\}_{k\in\mathbb{N}}$ a family of functions such that exists a constant c*

$$\|g_k\|_{L^q(\Omega)} \leq c, \quad g_k \to g \quad \text{a.e. in } \Omega \quad \text{as } k \to \infty.$$

Then $g_k \rightharpoonup g$ weakly in $L^q(\Omega)$.

Proof. See [19, Lemme 1.3. p. 12-13] □

Appendix B

Functional analysis

Lemma B.1 (Gronwall inequality in integral form)**.** *Let u and b be two non negative continuous functions on $[t_0, T]$ such that the inequality*

$$u(t) \leq C + \int_{t_0}^{t} b(s)u(s)ds \tag{B.1a}$$

holds for $t_0 \leq t \leq T$, where C is a positive constant. Then

$$u(t) \leq C \exp\left(\int_{t_0}^{T} b(s)\,\mathrm{d}s\right) \tag{B.1b}$$

for $t_0 \leq t \leq T$.

Lemma B.2 (Discrete Gronwall inequality)**.** *Let $a_i, b_i \geq 0$ for $i = 1, \ldots, N$. If u_m satisfies*

$$u_m \leq a_m + \sum_{i=1}^{m-1} b_i u_i \qquad \forall m = 1, \ldots, N \tag{B.2a}$$

then the estimate

$$u_m \leq a_m + \sum_{i=1}^{m-1} a_i \exp\left(\sum_{j=i}^{m-1} b_j\right) \qquad \forall m = 1, \ldots, N \tag{B.2b}$$

holds.

Lemma B.3 (Young Inequality with δ)**.** *For all $a, b \geq 0$ and $\delta > 0$ there holds*

$$ab \leq \delta a^p + c_{\delta,p} b^{p'} \tag{B.3}$$

where $p, p' > 1$ satisfy $(1/p) + (1/p') = 1$ and $c_{\delta,p} = (p')^{-1}(\delta p)^{-p'/p}$.

Remark B.4. An often used and special case is for $p = p' = \frac{1}{2}$, then yields for all $\delta > 0$

$$ab \leq \delta a^2 + \frac{1}{4\delta} b^2. \tag{B.4}$$

Lemma B.5 (Generalised Hölder Inequality)**.** *Let* $p_j \in [1, \infty]$ *and* $f_j \in L^{p_j}(\Omega)$ *for* $j = 1, \ldots, m$ *and set* $\frac{1}{r} := \sum_{j=1}^{m} \frac{1}{p_j}$*, then*

$$\prod_{j=1}^{m} f_j \in L^r(\Omega) \quad \textit{and satisfies} \quad \left\| \prod_{j=1}^{m} f_j \right\|_{L^r(\Omega)} \leq \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\Omega)}. \tag{B.5}$$

Theorem B.6 (Schauder fixed Point Theorem)**.** *Let* X *be a normed vector space and let* $K \subset X$ *be a non-empty, compact, and convex set. Then given any continuous mapping* $T : K \to K$

$$\textit{there exists } x \in K \quad \textit{such that} \quad T(x) = x. \tag{B.6}$$

Definition B.7 (Sobolev conjugate)**.** Take $n \geq 1$ and three real numbers j, k and p such that

$$0 \leq j < k, \quad 1 \leq p < n.$$

We set

$$\frac{1}{p^*} := \frac{1}{p} - \frac{k - j}{n}. \tag{B.7}$$

Then p^* is called the *Sobolev conjugate* of p.

We quote the general Sobolev embedding theorem from [25].

Theorem B.8 (The Sobolev embedding Theorem)**.** *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *with* $n \geq 1$ *and piecewise smooth boundary. Take three real numbers* j, k *and* p *according to the Definition B.7. Then, the following fundamental assertion hold. The embedding*

$$W^{k,p}(\Omega) \subset W^{j,q}(\Omega) \tag{B.8}$$

is continuous *for* $q \leq p^*$ *and* compact *for* $q < p^*$*.*

Proposition B.9 (Special Embedding in $L^p(\Omega)$ and Poincaré Inequality)**.** *Let* Ω *be a bounded domain in* $\mathbb{R}^3$ *with smooth boundary.*

Then the embedding

$$H^1(\Omega) \subset L^p(\Omega) \tag{B.9a}$$

is continuous for every $1 \leq p \leq 6$ *and compact for every* $1 \leq p < 6$*, where* j*,* k *and* p *are given by the above Definition and* p^* *denotes the Sobolev conjugate of* p*.*

There exists a constant M_Ω *with*

$$\|v\|_{L^p(\Omega)} \leq M_\Omega \|\nabla v\|_{L^2(\Omega)} \quad \textit{for every } v \in H_0^1(\Omega) \quad \textit{and} \quad 1 \leq p \leq 6. \tag{B.9b}$$

Proposition B.10 (Special Embedding in L^2)**.** *Let* Ω *be a bounded domain in* $\mathbb{R}^3$ *with smooth boundary.*

Then the embedding

$$W^{1,p}(\Omega) \subset L^2(\Omega) \tag{B.10}$$

is continuous for $p \geq 6/5$ *and compact for* $p > 6/5$*.*

Theorem B.11 (Embedding of parabolic spaces). *Let $\Omega \subset \mathbb{R}^3$ be a bounded and smooth domain, $(0,T)$ a finite time interval and set $Q := \Omega \times (0,T)$. Then we have for $m \geq 1$*

$$L^\infty(0,T;L^m(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \subset L^{q(m)}(Q) \quad \textit{where} \quad q(m) = \frac{2}{3}(m+3) \tag{B.11a}$$

and the embedding is continuous

$$\|v\|_{L^{q(m)}(Q)} \leq M_{\Omega,T,m} \left(\|v\|_{L^\infty(0,T;L^m(\Omega))} + \|\nabla v\|_{L^2(Q)} \right) \\ \textit{for every } v \in L^\infty(0,T;L^m(\Omega)) \cap L^2(0,T;H_0^1(\Omega)). \tag{B.11b}$$

Remark B.12. Again we need one special case of the above embedding theorem. So we set $m = 3$ and obtain the embedding

$$L^\infty(0,T;L^3(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \subset L^4(Q). \tag{B.12a}$$

Another important embedding result is obtained with $q(m) = 2$ and therefore $m = 10/3$

$$L^\infty(0,T;L^{10/3}(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \subset L^2(Q), \tag{B.12b}$$

where of course the corresponding estimates (B.11b) are holding.

Lemma B.13. *Take three Banach spaces B_0, B, B_1 with*

$$B_0 \subset B \subset B_1 \qquad \textit{and} \quad B_i \textit{ is reflexive for } i = 1,2, \\ \textit{and the injection } B_0 \to B \textit{ is compact.}$$

Then with $v \in B_0$ and $\forall \eta > 0$ exists a constant c_η with

$$\|v\|_B \leq \eta \|v\|_{B_0} + c_\eta \|v\|_{B_1}. \tag{B.13}$$

Proof. See [19, Lemme 5.1. p. 58]. □

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