Time Discretisation for a Class of Singular Phase Field Models

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Abstract

This paper is motivated by the work [2] which treats the mathematical analysis of a thermomechanical model describing phase transitions in terms of the entropy and order structure balance law. Our main purpose is to discretise this model in time, show the existence, uniqueness and boundedness of the order parameter χ for the discretised model. Finally, the paper gives a result on the convergence of the discretised model to the time continuous one.

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1 Introduction

1.1 Overview

The topic of this paper is the phenomena of phase transition. We consider binary systems of first and second order. The mathematical background was developed by Ginzburg and Landau in the 50's and is therefore called *Ginzburg-Landau-theory* [13]. Newer literature is from Fabrizio and Morro [11] and Fabrizio [10].

In a first order phase transition phenomena, as in the solid-liquid or liquid-vapour phase change, the phase transition occurs at a critical temperature ϑ_c. If the absolute temperature ϑ in the body is strictly greater than the critical temperature, then the minimum of the energy potential is attained in one of the pure phases, while if $\vartheta < \vartheta_c$ the minimum is attained in the other phase. In the case when $\vartheta = \vartheta_c$ the energy potential has two minima attained for the two phases, that is, the phase change may occur. On the other hand, in the case of second order phase transitions, the system behaves differently provided ϑ is greater or less than the critical temperature ϑ_c. Indeed, for high temperatures the energy potential has only one minimum, while for $\vartheta < \vartheta_c$ two minima are attained with the same values. This second behaviour is characteristic, for instance, of solid-solid phase transitions, ferromagnetism and superconductivity.

This paper presents a time discretisation of the model analysed in [2]. The model uses phase-field theory, which in terms of the temperature ϑ and a phase parameter χ includes the effects of micro-motions and micro-forces responsible for the phase transition. Frémond [12] suggested this new approach for models describing phase transitions by use of a generalisation of the principle of virtual power including the effects of micro-motions and micro-forces. For a more detailed derivation of the model itself we may confer to [5]. A similar theory was developed by Gurtin [14] for Ginzburg-Landau and Cahn-Hilliard equations, but in the isothermal case.

The main advantage of the model treated in this paper is that, once the problem is solved in a suitable sense, we can obtain directly the positivity of the absolute temperature, mainly caused by the presence of a logarithmic nonlinearity in the resulting system of partial differential equations. Therefore no maximum principle arguments have to be applied, which are difficult to set in a number of interesting situations.

1.2 Physical background

Our two-phase system is located in a smooth and bounded domain $\Omega \subset \mathbb{R}^3$ and we watch its evolution during a finite time interval $(0, T)$. We denote by Γ the boundary $\partial\Omega$. The thermomechanical equilibrium of the system is described in terms of a state variable and is governed by the free energy, while the dynamics reflect the presence of a pseudo-potential of dissipation. We do not consider mechanical effects. Thus the variables of the system are just the absolute temperature ϑ and a phase parameter χ, related to the proportion of one phase with respect to the other. In general, χ attains its physical admissible values in a range $[\chi_*, \chi^*]$ (for example $\chi \in [0, 1]$) representing the portion of the present phases for all time up to T. We will observe that this physical constraint is ensured by the model itself.

1.2.1 Evolution equations

The just introduced phase parameter χ can be interpreted as an *order parameter*. Thus it describes the change in the order structure of the thermomechanical system. For many materials the order structure below a critical temperature is greater than above. For example if we are thinking of liquid-solid phase transitions, then is the system below the melting temperature in a more ordered state. The phenomena is also characteristic for ferromagnetism: Above the Curie-temperature, the magnetic moments (also called *Weiss domains*) of the system are in a less ordered state and so the material is nonmagnetic.

Nevertheless, the parameter χ is a macroscopic parameter, the evolution of χ is governed by the micro-forces and micro-movements responsible for the phase transition at a microscopic level. Thus, the evolution of the order parameter χ can be derived from thermomechanical laws, as a balance equation for micro-forces. The balance conditions are read in the following way

$$B - \operatorname{div} \mathbf{H} = 0 \quad \text{in } \Omega \times (0,T), \quad \mathbf{H} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \times (0,T), \tag{1.1}$$

where $\mathbf{n}$ is the outward normal to the boundary Γ. In addition B can be interpreted as an energy density per units of concentration χ as well as $\mathbf{H}$ represents an energy flux vector. Note that with the homogeneous Neumann boundary conditions we assume that no external work is carried out to the system.

For the evolution of the temperature ϑ we will use the first and second principle of thermodynamics. We use a rescaled energy balance in which higher order dissipative contributions are neglected by means of the small perturbations assumption.

We obtain a balance, which could be considered as an “entropy equation”, because it describes the evolution of the entropy s of the system in terms of the entropy flux $\mathbf{Q}$ and an external source $R(\vartheta)$, possibly depending on the temperature ϑ and maybe singular. Thus the following equation holds

$$s_t + \operatorname{div} \mathbf{Q} = R(\vartheta) \quad \text{in } \Omega \times (0,T). \tag{1.2}$$

We may demand boundary conditions on the entropy flux. If no flux is assumed through the boundary, then holds $\mathbf{Q} \cdot \mathbf{n} = 0$ on $\Gamma \times (0,T)$. Later we will ask for the value of the temperature on the boundary.

1.2.2 Energy functionals

We specify the involved physical quantities with the help of two energy functionals: on the one hand, the *free energy* Ψ, depending on the state variables and accounting for the thermomechanical equilibrium of the system; on the other hand, the *pseudo-potential of dissipation* Φ, defined for the dissipative variables responsible for the evolution of the system. For details on these functionals we refer to Moreau [19].

Free energy Our state variables are the absolute temperature ϑ, the order parameter χ and its gradient $\nabla\chi$. The thermodynamical laws tell us that the free energy is a concave function with respect to the temperature while there are no constraints concerning the dependence on the other variables. Therefore we choose the functional Ψ of the following form

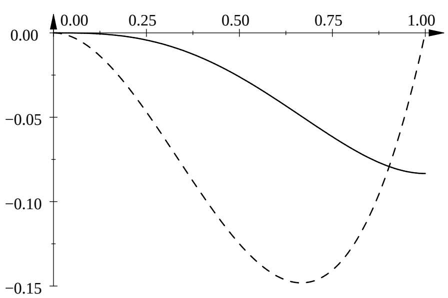
$$\Psi(\vartheta, \chi, \nabla\chi) = -\frac{c_0}{2}\vartheta^2 + F(\chi)\vartheta_c + G(\chi)\vartheta + \frac{\nu}{2}|\nabla\chi|^2, \tag{1.3}$$

where the constants c_0, ν are positive and $\vartheta_c > 0$ represents the already introduced critical temperature for the phase transition. Indeed we note that the purely caloric part of the free energy $-(c_0/2)\vartheta^2$ is concave.

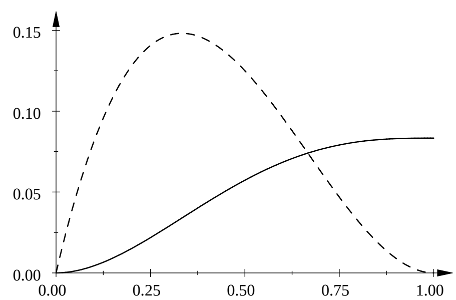
In addition the functions F and G characterise the behaviour of the phase transition. A first order phase transition, for instance a liquid-solid or a vapour-liquid system, can be described as follows

$$F(\chi) = \frac{\chi^4}{4} - \frac{\chi^3}{3}, \quad G(\chi) = \frac{\chi^4}{4} - \frac{2\chi^3}{3} + \frac{\chi^2}{2}. \tag{1.4}$$

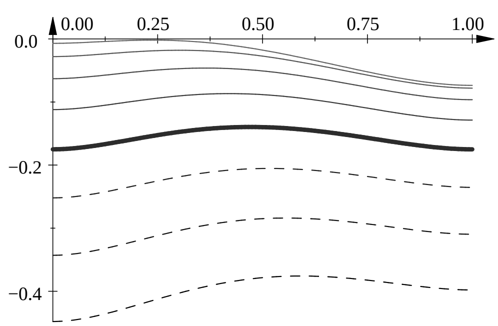
We want to stress out that for low temperatures $\vartheta < \vartheta_c$ the minimum of the free energy Ψ is attained in the pure first phase $\chi = 0$ and for high temperatures $\vartheta > \vartheta_c$ the minimum is attained in the pure second phase $\chi = 1$. The physical admissible range of χ is in this case $[0, 1]$. However in the equilibrium case $\vartheta = \vartheta_c$ there exist two minima in the two phases (cf. Figure 1).



(a) F and F' (dashed)



(b) G and G' (dashed)



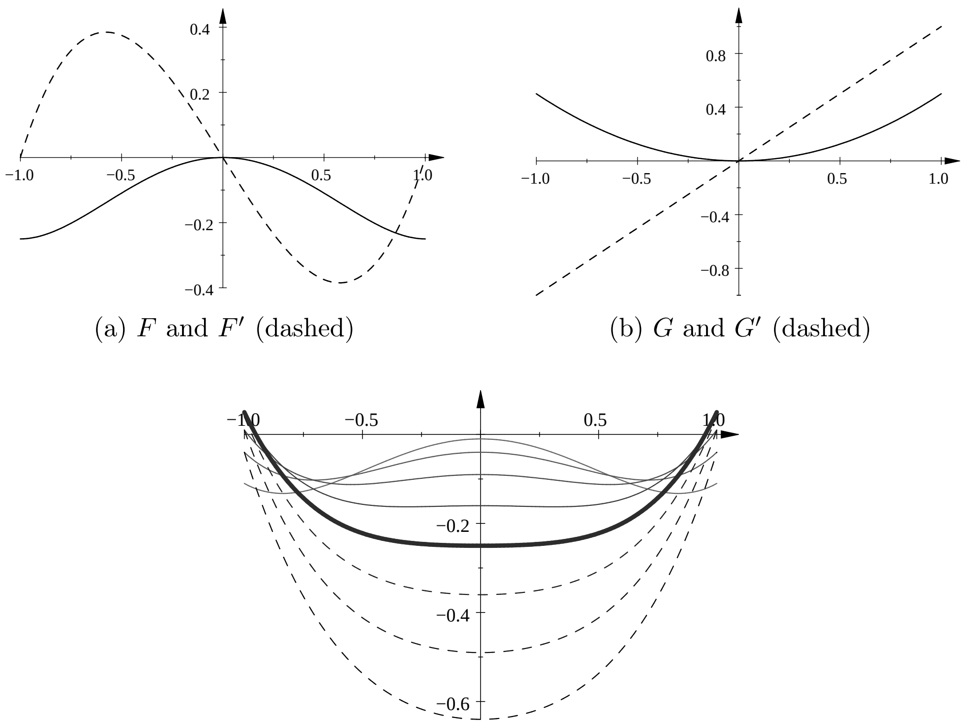
(c) Free energy Ψ for ϑ from 0.2 up to 1.6 ($\vartheta < \vartheta_c$ solid, $\vartheta = \vartheta_c$ bold, $\vartheta > \vartheta_c$ dashed)

Figure 1: Free energy for first order phase transitions ($c_0 = 0.35, \nu = 0.5, \vartheta_c = 1$)

Concerning a second order phase transition, for example superconductivity or ferromagnetism, F and G can be written as

$$F(\chi) = \frac{\chi^4}{4} - \frac{\chi^2}{2}, \quad G(\chi) = \frac{\chi^2}{2}. \tag{1.5}$$

Differently to the first order phase transition, the free energy Ψ attains two minima for low temperature $\vartheta < \vartheta_c$ in the mixed phase region once for $\chi \in (-1, 0)$ and once for $\chi \in (0, 1)$. The physical admissible values for χ are here $[-1, 1]$. Although, for temperatures $\vartheta \geq \vartheta_c$ there is again one minimum attained in $\chi = 0$ (cf. Figure 2).



(a) F and F' (dashed)

(b) G and G' (dashed)

(c) Free energy Ψ for ϑ from 0.2 up to 1.6
($\vartheta < \vartheta_c$ solid, $\vartheta = \vartheta_c$ bold, $\vartheta > \vartheta_c$ dashed)

Figure 2: Free energy for second order phase transitions ($c_0 = 0.5, \nu = 0.1, \vartheta_c = 1$)

Pseudo-potential of dissipation We introduce a second functional Φ that depends on the dissipative variables χ_t and $\nabla\vartheta$. We choose χ_t, because it is related to microscopic velocities which are responsible for the phase transition, while $\nabla\vartheta$ is concerned with the head flux. In addition, the pseudo-potential of dissipation Φ is nonnegative, convex with respect to the dissipative variables, and it attains its minimum 0 for a null dissipation, that is, when $(\chi_t, \nabla\vartheta) = (0, \mathbf{0})$. Thus we prescribe

$$\Phi(\chi_t, \nabla\vartheta) = \frac{\mu}{2} |\chi_t|^2 + \frac{\lambda}{2\vartheta} |\nabla\vartheta|^2 \tag{1.6}$$

with μ and λ denoting positive coefficients.

1.2.3 System of partial differential equations

All the quantities B, $\mathbf{H}$, s and $\mathbf{Q}$ can be recovered by the *constitutive relations* from the free energy, for nondissipative contributions and the pseudo-potential of dissipation, for the dissipative parts. We have

$$B = \frac{\partial \Psi}{\partial \chi} + \frac{\partial \Phi}{\partial \chi_t} = \vartheta_c F'(\chi) + \vartheta G'(\chi) + \mu \chi_t \tag{1.7a}$$

$$\mathbf{H} = \frac{\partial \Psi}{\partial (\nabla \chi)} = \nu \nabla \chi \tag{1.7b}$$

$$s = -\frac{\partial \Psi}{\partial \vartheta} = c_0 \vartheta - G(\chi) \tag{1.7c}$$

$$\mathbf{Q} = -\frac{\partial \Phi}{\partial (\nabla \vartheta)} = -\frac{\lambda}{\vartheta} \nabla \vartheta = -\lambda \nabla \log \vartheta. \tag{1.7d}$$

We want to point out that the choice of the free energy (1.3) leads to a linear contribution for the temperature in (1.7c). This preserves sufficient regularity on the solution. In addition that the term $-(c_0/2)\vartheta^2$ could be seen as a first order approximation of the following more general form of the energy potential

$$\Psi(\vartheta, \dots) = -c_0 \vartheta \log \vartheta + \dots .$$

In this case, the entropy s would be related to the temperature ϑ through a logarithmic nonlinearity. On the other hand, a logarithmic nonlinearity forcing ϑ to be strictly positive is present in our expression (1.7d) for the entropy flux $\mathbf{Q}$.

1.2.4 Initial-boundary value problem

Now, combining the constitutive relations (1.7a)-(1.7d) with the evolution equation for the energy (1.1) and the evolution equation for the entropy (1.2) leads to the following system of partial differential equations with initial and boundary conditions

$$c_0 \vartheta_t - G'(\chi)\chi_t - \lambda \Delta \log \vartheta = R(x, t, \vartheta) \tag{1.8a}$$

$$\mu \chi_t - \nu \Delta \chi + F'(\chi)\vartheta_c + G'(\chi)\vartheta = 0 \tag{1.8b}$$

which is addressed in $Q := \Omega \times (0, T)$. We fix Dirichlet boundary conditions in $\Gamma \times (0, T)$ for the temperature ϑ and Neumann homogeneous boundary conditions for the phase parameter χ

$$\log \vartheta = \log \vartheta_\Gamma, \quad \partial_n \chi = 0, \tag{1.9}$$

where ∂_n is the external normal derivative. Finally, initial conditions are

$$\vartheta(0) = \vartheta_0, \quad \chi(0) = \chi_0 \quad \text{in } \Omega. \tag{1.10}$$

For the sake of simplicity, in the mathematical analysis performed in subsequent sections we will take the physical constants $c_0, \lambda, \mu, \nu, \vartheta_c$ all equal to 1.

2 Main results

2.1 The continuous problem

In the sequel, Ω is a bounded open set in $\mathbb{R}^3$, whose boundary Γ is assumed to be of class C^2. Moreover, ∂_n is the outward normal derivative on Γ. Given a finite final time T, we set for convenience

$$Q_t := \Omega \times (0,t) \quad \text{for every } t \in (0,T] \quad \text{and} \quad Q := Q_T. \tag{2.1}$$

Four constants $\vartheta_*, \vartheta^*, \chi_*, \chi^* \in \mathbb{R}$ are given such that

$$0 < \vartheta_* \le 1 \le \vartheta^* \quad \text{and} \quad \chi_* < \chi^* \tag{2.2a}$$

and four functions F, G, β and π

$$F, G : \mathbb{R} \to \mathbb{R}, \quad \beta : Q \times (0,\infty) \to \mathbb{R}, \quad \text{and} \quad \pi : Q \times \mathbb{R} \to \mathbb{R}$$

must satisfy

$$F, G \in C^2(\mathbb{R}), \quad F \text{ is bounded from below and } G \text{ is nonnegative} \tag{2.2b}$$

$$F', G' < 0 \text{ in } (-\infty, \chi_*), \quad \text{and} \quad F', G' > 0 \text{ in } (\chi^*, \infty) \tag{2.2c}$$

$$\beta \text{ is Lipschitz continuous in } Q \times [\delta, 1/\delta] \text{ for every } \delta \in (0,1) \tag{2.2d}$$

$$\beta_{,x}, \beta_{,t}, \beta' \text{ and } \pi \text{ are Carathéodory functions with the notation} \tag{2.2e}$$

$$\beta_{,x}(x,t,r) := \nabla\beta(x,t,r), \quad \beta_{,t} := \partial_t\beta(x,t,r), \quad \beta'(x,t,r) := \partial_r\beta(x,t,r) \tag{2.2f}$$

$$0 \le \beta'(x,t,r) \le \beta_1(r) \text{ for a.e. } (x,t) \in Q, \forall r \in \mathbb{R} \text{ and some } \beta_1 \in C^0(0,\infty) \tag{2.2g}$$

$$\begin{aligned} &|\beta_{,x}(x,t,r)| + |\beta_{,t}(x,t,r)| \le M_\beta(1 + |\beta(x,t,r)|) \\ &\qquad \text{for a.a. } (x,t) \in Q, \forall r \in \mathbb{R} \text{ and some } M_\beta \in [0,\infty) \end{aligned} \tag{2.2h}$$

$$\beta(x,t,1) = 0 \text{ for every } (x,t) \in Q \tag{2.2i}$$

$$\pi(x,t,r) \text{ is uniformly Lipschitz continuous in } r \text{ for a.a. } (x,t) \in Q \text{ with constant } L_\pi \tag{2.2j}$$

$$\pi(\cdot,\cdot,0) = \pi_0(\cdot,\cdot) \in L^2(Q) \tag{2.2k}$$

Remark 2.1. The bound of β' in (2.2g) and again (2.2i) imply that

$$|\beta(x,t,r)| \le \beta_0(r) := \left| \int_1^r \beta_1(s)\,\mathrm{d}s \right| \quad \text{for all } (x,t,r) \in Q \times (0,\infty). \tag{2.3}$$

Therefore, by (2.2h) follows that even $\beta_{,x}$ and $\beta_{,t}$ are satisfying an analogous bound and in fact (2.2d) follows from the other assumptions.

Notation 2.2. Let I be a real interval and $\psi : Q \times I \to \mathbb{R}$ be a Carathéodory function. The same symbol ψ is used to denote the operator acting on measurable functions on Q as follows: if $v : Q \to I \subset \mathbb{R}$ is measurable

$$\psi(v) \quad \text{denotes the function} \quad (x,t) \mapsto \psi(x,t,v(x,t)), \quad (x,t) \in Q. \tag{2.4a}$$

Note that $\psi(v)$ is measurable due to the Carathéodory assumption on ψ. Similar definitions and symbols are used for functions depending on the space variable. Namely, if $v : \Omega \to I \subset \mathbb{R}$ is measurable

$$\psi(t,v) \quad \text{denotes the function} \quad x \mapsto \psi(x,t,v(x)), \quad x \in \Omega \tag{2.4b}$$

for a.a. $t \in (0,T)$. As well as if $\phi : \Omega \times I \to \mathbb{R}$ is a Carathéodory function

$$\phi(v) \quad \text{denotes the function} \quad x \mapsto \phi(x, v(x)), \quad x \in \Omega. \tag{2.4c}$$

In addition, the abbreviation

$$H_n^2(\Omega) := \left\{ v \in H^2(\Omega) : \partial_n v = 0 \text{ on } \Gamma \right\} \tag{2.5}$$

is used.

There are given three functions ϑ_Γ, ϑ_0 and χ_0 such that

$$\vartheta_\Gamma \in L^2(0,T : H^{1/2}(\Gamma)) \cap H^1(0,T; H^{-1/2}(\Gamma)), \quad \vartheta_* \le \vartheta_\Gamma \le \vartheta^* \quad \text{a.e. on } \Gamma \times (0,T) \tag{2.6a}$$

$$\vartheta_0 \in L^2(\Omega), \quad \vartheta_* \le \vartheta_0 \le \vartheta^* \quad \text{a.e. in } \Omega \tag{2.6b}$$

$$\chi_0 \in H^1(\Omega), \quad \chi_* \le \chi_0 \le \chi^* \quad \text{a.e. in } \Omega \tag{2.6c}$$

where $\vartheta_*, \vartheta^*, \chi_*$ and χ^* are introduced in (2.2a).

Definition 2.3 (Solution of the continuous problem)**.** The triplet (ϑ, χ, ξ) is called a solution of the continuous problem if it fulfils

$$\vartheta \in L^\infty(0,T; L^2(\Omega)), \quad \vartheta > 0 \quad \text{a.e. in } Q, \quad \text{and} \quad \ln\vartheta \in L^2(0,T; H^1(\Omega)) \tag{2.7a}$$

$$\chi \in L^2(0,T; H_n^2(\Omega)) \cap H^1(0,T; L^2(\Omega)) \tag{2.7b}$$

$$G(\chi), F'(\chi), G'(\chi) \in L^2(Q) \tag{2.7c}$$

$$\xi \in L^2(Q) \tag{2.7d}$$

$$\partial_t(\vartheta - G(\chi)) \in L^2(0,T; H^{-1}(\Omega)) \tag{2.7e}$$

$$\partial_t(\vartheta - G(\chi)) - \Delta \ln\vartheta + \xi = \pi(\vartheta) \quad \text{in } L^2(0,T; H^{-1}(\Omega)) \quad \text{and} \quad \xi = \beta(\vartheta) \tag{2.7f}$$

$$\partial_t\chi - \Delta\chi + F'(\chi) + G'(\chi)\vartheta = 0 \quad \text{a.e. in } Q \tag{2.7g}$$

$$\ln\vartheta = \ln\vartheta_\Gamma \quad \text{a.e. on } \Gamma \times (0,T) \tag{2.7h}$$

$$(\vartheta - G(\chi))(0) = \vartheta_0 - G(\chi_0) \quad \text{and} \quad \chi(0) = \chi_0 \quad \text{a.e. in } \Omega. \tag{2.7i}$$

Remarks 2.4. (i) Even though ξ is a known function of ϑ, we refer to the triplet (ϑ, χ, ξ) instead of the pair (ϑ, χ), when we speak of a solution.

(ii) Moreover, we note that (2.7a) and (2.7c) yield $G'(\chi)\vartheta \in L^2(0,T; L^1(\Omega))$ and by comparison to (2.7g), even $G'(\chi)\vartheta \in L^2(Q)$.

(iii) We point out that the first condition in (2.7i) reduces to $\vartheta(0) = \vartheta_0$ whenever one knows $G(\chi) \in C^0([0,T]; H^{-1}(\Omega))$.

(iv) Actually, some additional smoothness for $G(\chi)$ as well as for $F'(\chi)$ and $G'(\chi)$ surely holds if the nonlinearities satisfy some growth conditions, thanks to (2.7b) or whenever χ is bounded and our existence result stated below ensures such a property.

(v) A homogeneous Neumann boundary condition for χ is entailed by (2.7b) due to the introduced space $H_n^2(\Omega)$.

In [2] we find the analysis with existence, uniqueness and boundedness results of the problem in detail. As a consequence of the maximum principle for χ (cf. [2, Theorem 2.4]), we can replace F and G by new Lipschitz continuous functions, still termed F and G, satisfying

$$|F'| \leq L_F \quad \text{and} \quad |G'| \leq L_G. \tag{2.8}$$

Indeed we can arbitrarily modify F and G outside $[\chi_*, \chi^*]$.

2.2 Results for the discrete problem

Definition 2.5 (Partition)**.** A *partition* $\mathcal{P}$ of the interval $[0,T]$ is defined as the ordered set

$$\mathcal{P} := \{t_0 = 0, t_1, \ldots, t_{N-1}, t_N = T\}, \text{ where } t_0 < t_1 < \cdots < t_N. \tag{2.9}$$

The size of every subinterval is denoted by $\tau_i = t_i - t_{i-1}$ and the *diameter* of the partition is $\tau := \max_i \tau_i$.

The partition $\mathcal{P}$ is said to be *uniform*, if $\min_i \tau_i \geq \sigma\tau$ for a fixed σ with $0 < \sigma \leq 1$.

Notation 2.6. Let $\{\vartheta^i\}_{i=1}^N$ be a set of elements in a Banach space V and $\mathcal{P}$ a partition with N subintervals, then the vector $\vartheta^{\mathcal{P}}$ denotes

$$\vartheta^{\mathcal{P}} := (\vartheta^1, \ldots, \vartheta^N) \in V^N. \tag{2.10}$$

Definition 2.7 (Step approximations)**.** Let $\alpha : Q \to \mathbb{R}$ be a locally integrable function and $\mathcal{P}$ denotes a partition with N subintervals. Then two sets of interpolating points are defined

$$\alpha^i(x) := \alpha(x, t_i) \quad \text{for } i = 0, \ldots, N \quad \text{and} \quad \overline{\alpha}^i(x) := \frac{1}{\tau_i}\int_{t_{i-1}}^{t_i} \alpha(x,s)\,\mathrm{d}s \quad \text{for } i = 1, \ldots, N. \tag{2.11}$$

In addition $\overline{\alpha}^0(x) := 0$.

Remark 2.8. This notation is used for β

$$\beta^i(x,r) := \beta(x, t_i, r).$$

In addition the notation with the bar is used for π and ϑ_Γ. In detail we define for $1 \leq i \leq N$

$$\overline{\pi}^i(\cdot,\cdot) = \frac{1}{\tau^i}\int_{\tau^{i-1}}^{\tau^i} \pi(\cdot, s, \cdot)\,\mathrm{d}s \quad \text{on } \Omega \times \mathbb{R} \text{ and } \quad \overline{\vartheta}^i_\Gamma(\cdot) = \frac{1}{\tau^i}\int_{\tau^{i-1}}^{\tau^i} \vartheta_\Gamma(\cdot, s)\,\mathrm{d}s \quad \text{on } \Gamma.$$

Definition 2.9 (Step and linear interpolations)**.** Take a uniform partition $\mathcal{P}$ and vectors $\vartheta^{\mathcal{P}}$ and $\chi^{\mathcal{P}}$. First, the definitions

$$\vartheta_\tau(x,t) = \vartheta^i(x) \quad \text{and} \quad \chi_\tau(x,t) = \chi^i(x) \quad \text{for a.a. } (x,t) \in \Omega \times (t_{i-1}, t_i) \tag{2.12a}$$

denote the *step interpolations* of $\vartheta^{\mathcal{P}}$ and $\chi^{\mathcal{P}}$. The *linear interpolation* of $\vartheta^{\mathcal{P}}$ and $\chi^{\mathcal{P}}$ are defined by

$$\begin{aligned}\widehat{\vartheta}_\tau(x,t) &= \frac{t-t_{i-1}}{\tau_i}\vartheta^i(x) + \frac{t_i - t}{\tau^i}\vartheta^{i-1}(x) \qquad \text{for a.a. } (x,t)\in\Omega\times(t_{i-1},t_i)\\ \widehat{\chi}_\tau(x,t) &= \frac{t-t_{i-1}}{\tau_i}\chi^i(x) + \frac{t_i - t}{\tau^i}\chi^{i-1}(x) \qquad \text{for a.a. } (x,t)\in\Omega\times(t_{i-1},t_i).\end{aligned} \tag{2.12b}$$

As well as for the vector $\beta^{\mathcal{P}} = (\beta^1,\ldots,\beta^N)$ the *step* and *linear interpolation*

$$\beta_\tau(x,t,r) = \beta^i(x,r) \quad \text{and} \quad \widehat{\beta}_\tau(x,t,r) = \frac{t-t_{i-1}}{\tau_i}\beta^i(x,r) + \frac{t_i - t}{\tau^i}\beta^{i-1}(x,r) \tag{2.12c}$$

are defined for a.a. $(x,t,r)\in\Omega\times(t_{i-1},t_i)\times\mathbb{R}$. Next, $\overline{\pi}^{\mathcal{P}} = (\overline{\pi}^1,\ldots,\overline{\pi}^N)$ and $\overline{\vartheta}_\Gamma^{\mathcal{P}} = (\overline{\vartheta}_\Gamma^1,\ldots,\overline{\vartheta}_\Gamma^N)$ defined by

$$\begin{aligned}&\overline{\pi}_\tau(x,t,r) = \overline{\pi}^i(x,r) \qquad \text{for a.a. } (x,t,r)\in\Omega\times(t_{i-1},t_i)\times\mathbb{R}\\ &\overline{\vartheta}_{\Gamma,\tau} = \overline{\vartheta}_\Gamma^i(x) \qquad \text{for a.a. } (x,t)\in\Gamma\times(t_{i-1},t_i)\end{aligned} \tag{2.12d}$$

are the *step interpolations* respecting the mean values introduced above.

Remark 2.10. The linear interpolations of ϑ and χ are useful for representing the backward approximations of their time derivative. Indeed, for $t\in(t_{i-1},t_i)$ we have

$$\partial_t\widehat{\vartheta}_\tau = \frac{\vartheta^i - \vartheta^{i-1}}{\tau_i} \quad \text{and} \quad \partial_t\widehat{\chi}_\tau = \frac{\chi^i-\chi^{i-1}}{\tau_i}. \tag{2.13}$$

For defining the solution of the associated time-discrete problem, we replace the time derivatives $\partial_t\vartheta$ and $\partial_t\chi$ in the continuous problem (2.7f), (2.7g) by their backward approximations as well as the right hand side and boundary data by their step approximations.

Definition 2.11 (Solution of the discrete problem)**.** Let $\mathcal{P}$ denote a uniform partition of the interval $[0,T]$. The triplet $(\vartheta^{\mathcal{P}},\chi^{\mathcal{P}},\xi^{\mathcal{P}})$ is called *solution of the discrete problem* if it satisfies for all $1\le i\le N$

$$\vartheta^i\in L^2(\Omega), \quad \ln\vartheta^i\in H^1(\Omega) \qquad \vartheta^i>0 \quad \text{a.e. in } Q \tag{2.14a}$$

$$\chi^i\in H^2_n(\Omega) \quad \text{and} \quad \xi^i\in L^2(\Omega) \tag{2.14b}$$

$$F'(\chi^i), G'(\chi^i)\in L^2(\Omega) \tag{2.14c}$$

$$\frac{\vartheta^i-\vartheta^{i-1}}{\tau^i} - G'(\chi^i)\frac{\chi^i-\chi^{i-1}}{\tau^i} - \Delta\ln\vartheta^i + \xi^i = \overline{\pi}^i(\vartheta^{i-1}) \quad \text{and} \quad \xi^i = \beta^i(\vartheta^i) \tag{2.14d}$$

$$\frac{\chi^i-\chi^{i-1}}{\tau^i} - \Delta\chi^i + F'(\chi^i) + G'(\chi^i)\vartheta^{i-1} = 0 \tag{2.14e}$$

and the initial boundary conditions

$$\ln\vartheta^i = \ln\overline{\vartheta}_\Gamma^i \quad \text{a.e. on } \Gamma \tag{2.14f}$$

$$\vartheta^0 = \vartheta_0 \quad \text{and} \quad \chi^0 = \chi_0 \quad \text{a.e. in } \Omega. \tag{2.14g}$$

Theorem 2.12 (Existence, uniqueness and boundedness of χ). *Let the assumptions* (2.2) *be fulfilled as well as the initial and boundary data satisfy the regularity conditions* (2.6)*. Then there exists a unique triplet* $(\vartheta^{\mathcal{P}}, \chi^{\mathcal{P}}, \xi^{\mathcal{P}})$ *solving the associated discrete problem* (2.14) *according to Definition 2.11. Moreover, the solution* $(\vartheta^{\mathcal{P}}, \chi^{\mathcal{P}}, \xi^{\mathcal{P}})$ *fulfils the inequalities*

$$\chi_* \le \chi^i \le \chi^* \quad \text{for all } 1 \le i \le N \text{ and a.e. in } \Omega. \tag{2.15}$$

In particular, each coordinate of $\chi^{\mathcal{P}}$ *is bounded.*

The proof of this theorem is provided in Section 4.

Theorem 2.13 (Stability result). *There exists a constant* c *such that for every* $\tau > 0$*, sufficiently small,* $(\vartheta^{\mathcal{P}}, \chi^{\mathcal{P}}, \xi^{\mathcal{P}})$ *is a discrete solution with*

$$\begin{aligned}
&\|\vartheta_\tau\|^2_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;W^{1,4/3}(\Omega))} + \left\|\widehat{\vartheta}_\tau\right\|_{H^1(0,T;H^{-1}(\Omega))} + \tau \left\|\widehat{\vartheta}_\tau\right\|^2_{H^1(0,T;L^2(\Omega))} \\
&+ \|\chi_\tau\|^2_{L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega))} + \left\|\widehat{\chi}_\tau\right\|^2_{H^1(0,T;L^2(\Omega))} + \tau \left\|\widehat{\chi}_\tau\right\|^2_{H^1(0,T;H^1(\Omega))} \\
&+ \sup_{t\in[0,T]} \int_\Omega (\vartheta_\tau(t)(\ln \vartheta_\tau(t) - 1) + 1) + \|\ln \vartheta_\tau\|^2_{L^2(0,T;H^1(\Omega))} + \|\xi_\tau\|_{L^2(0,T;L^2(\Omega))} \le c.
\end{aligned} \tag{2.16}$$

We want to remark that $\xi_\tau = \beta_\tau(\vartheta_\tau)$ *in consequence of* (2.14d).

Pointwise convergence can be established on the basis of the previous theorem.

Theorem 2.14 (Convergence to continuous solution). *Let initial and boundary data be given satisfying* (2.6)*. Take a sequence of partitions* $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ *with* $\tau_n \to 0$*, where* τ_n *is the diameter of the partition* $\mathcal{P}_n$*. Then there exists a subsequence* $\{n_k\}_{k\in\mathbb{N}}$ *such that*

$$(\vartheta_{\tau_{n_k}}, \chi_{\tau_{n_k}}, \xi_{\tau_{n_k}}) \to (\vartheta, \chi, \xi) \quad \text{a.e. in } Q \quad \text{as} \quad k \to \infty. \tag{2.17a}$$

In detail the following convergences hold

$$\vartheta_{\tau_{n_k}} \to \vartheta \qquad \text{strongly in } L^2(0,T;L^2(\Omega)). \tag{2.17b}$$

$$\chi_{\tau_{n_k}} \to \chi \qquad \text{strongly in } L^2(0,T;H^1(\Omega)). \tag{2.17c}$$

$$\xi_{\tau_{n_k}} \to \xi \qquad \text{a.e. in } Q \text{ and weakly in } L^2(Q), \tag{2.17d}$$

where (ϑ, χ, ξ) *is a solution of the associated continuous problem.*

The previous two theorems are proved in Section 5.

Remark 2.15.

1. Theorem 2.12 and 2.14 reproduce the existence result of [2].
2. Moreover, in [2] uniqueness is shown under the given assumptions, especially (2.2j).
3. In [22] a result on the continuous dependence on the data of the continuous problem can be found.

3 Tools

3.1 Yosida approximation

Definition 3.1 (Regularization of ln)**.** The resolvent $\rho_\varepsilon : \mathbb{R} \to \mathbb{R}_+$ of the logarithm is the solution of the transcendental equation

$$\rho_\varepsilon(r) + \varepsilon \ln \rho_\varepsilon(r) = r. \tag{3.1a}$$

Then, the *Yosida approximation* is given by

$$\ln_\varepsilon r := \frac{r - \rho_\varepsilon(r)}{\varepsilon}. \tag{3.1b}$$

Later, we use the slightly altered function $\mathrm{Ln}_\varepsilon : \mathbb{R} \to \mathbb{R}$

$$\mathrm{Ln}_\varepsilon r := \varepsilon r + \ln_\varepsilon r \quad \text{and its primitive} \quad \mathcal{L}_\varepsilon(r) := \int_1^r \mathrm{Ln}_\varepsilon s \,\mathrm{d}s = \frac{\varepsilon}{2}(r^2 - 1) + \int_1^r \ln_\varepsilon s \,\mathrm{d}s. \tag{3.1c}$$

Proposition 3.2 (Properties of the resolvent ρ_ε)**.** *The resolvent ρ_ε defined in* (3.1a) *admits the representation*

$$\rho_\varepsilon(r) = \exp\left(\frac{r}{\varepsilon} - \mathrm{W}\left(\frac{1}{\varepsilon} e^{\frac{r}{\varepsilon}}\right)\right), \tag{3.2}$$

where W, *called* Lambert W function, *fulfils the equation $z = \mathrm{W}(z)e^{\mathrm{W}(z)}$. In the representation we use the first real branch of* W. *Additionally, ρ_ε is for every fixed $\varepsilon > 0$ a strictly monotone, convex and positive function.*

Proof. We substitute $\rho_\varepsilon = e^{\phi_\varepsilon}$ in (3.1a) and get $e^{\phi_\varepsilon} + \varepsilon\phi_\varepsilon = r$. Now, setting $t = \frac{r}{\varepsilon} - \phi_\varepsilon$ and some transformations result in $te^t = -\frac{1}{\varepsilon} e^{\frac{r}{\varepsilon}}$. With the definition of Lambert's W function and substituting t back, we obtain $\phi_\varepsilon(r) = \frac{r}{\varepsilon} - \mathrm{W}\left(\frac{1}{\varepsilon} e^{\frac{r}{\varepsilon}}\right)$. This equation gives the representation by definition of ϕ_ε.

The positivity of ρ_ε for fixed ε directly follows from the representation (3.2), because we choose the real branch of W. We calculate the first derivative implicitly

$$\rho_\varepsilon'(r) = \frac{\rho_\varepsilon(r)}{\rho_\varepsilon + \varepsilon} = 1 - \frac{\varepsilon}{\rho_\varepsilon(r) + \varepsilon} > 0$$

as ρ_ε is positive. We remark that obviously also $\rho_\varepsilon'(r) \le 1$. Further differentiating leads to

$$\rho_\varepsilon''(r) = \frac{\rho_\varepsilon'(r)}{\rho_\varepsilon(r) + \varepsilon}\left(1 - \frac{\rho_\varepsilon^2(r)}{(\rho_\varepsilon(r) + \varepsilon)^2}\right) \ge 0$$

by the same argument. Thus ρ_ε is also convex. □

Proposition 3.3 (Properties of the approximations $\ln_\varepsilon$ and Ln_ε)**.** *We have*

$$\ln_\varepsilon^{-1}(s) = e^s + \varepsilon s \quad \textit{for every } s \in \mathbb{R} \tag{3.3a}$$

$$\frac{\ln r}{1 + \varepsilon} \le \ln_\varepsilon r \le \ln r \quad \textit{for every } r \ge 1 \tag{3.3b}$$

$$\mathrm{Ln}_\varepsilon'(r) \ge 1 \quad \textit{for every } r \le 1 \quad \textit{and} \quad \mathrm{Ln}_\varepsilon'(r) \ge \frac{1}{2r} \quad \textit{for every } r > 1 \tag{3.3c}$$

We set $l_* := \min\{0, \ln\vartheta_*\}$ *and* $l^* := \max\{0, \ln\vartheta^*\}$*, then we have*

$$l_* \le \ln_\varepsilon r \le l^* \quad \textit{for every } r \in [\vartheta_*, \vartheta^*]. \tag{3.3d}$$

Proof. We refer to [2, Proposition 3.2] where details of the proof are given. □

Remark 3.4. $\mathcal{L}_\varepsilon(r)$ is a convex function, which is bounded from below uniformly with respect to ε, since $\ln_\varepsilon 1 = 0$.

3.2 Time approximation

Definition 3.5 (Translation operator)**.** We define for $k \in \mathbb{Z}$ the *translation operator* $\mathcal{T}_k$ acting on approximations α_τ on a partition $\mathcal{P}$. Take $t \in (t_{i-1}, t_i)$, then $\mathcal{T}_k$ is defined by

$$\mathcal{T}_k[\alpha_\tau](t) := \begin{cases} \alpha(0) & \text{for } k+i \le 0 \\ \alpha\left(\left(\frac{t-t_{i-1}}{t_i}\right)\tau_{i-k} + t_{i-k-1}\right) & \text{for } 0 < k+i < N \\ \alpha(t_n) & \text{for } k+i \ge N. \end{cases} \tag{3.4}$$

Lemma 3.6. *Let* $\mathcal{P}$ *denote a uniform partition with diameter* τ*. In addition* $\varphi : \mathbb{R} \to [0, \infty)$ *is a convex function and* $\alpha : Q \to \mathbb{R}$ *is locally integrable, then*

$$(\varphi \circ \overline{\alpha}_\tau)(x,t) \le \overline{(\varphi \circ \alpha)}_\tau(x,t) \quad \textit{for all } (x,t) \in Q, \tag{3.5}$$

in which $\circ$ *denotes the composition of two functions* $f \circ g := f(g)$*.*

Proof. Fix $x \in \Omega$ and take $t \in (t_{i-1}, t_i)$ for some $1 \le i \le N$. Then

$$\begin{aligned} \varphi(\overline{\alpha}^i(x)) &= \varphi\left(\frac{1}{\tau_i}\int_{t_{i-1}}^{t_i} \alpha(x,s)\,\mathrm{d}s\right) = \varphi\left(\int_0^1 \alpha(x, \tau_i\widetilde{s} + t_{i-1})\,\mathrm{d}\widetilde{s}\right) \\ &\le \int_0^1 \varphi\left(\alpha(x, \tau_i\widetilde{s} + t_{i-1})\right)\mathrm{d}\widetilde{s} = \frac{1}{\tau_i}\int_{t_{i-1}}^{t_i} \varphi\left(\alpha(x,s)\right)\,\mathrm{d}s = \overline{\varphi(\alpha)}^i(x,t), \end{aligned}$$

where the estimate is a consequence of Jensen's inequality. □

Proposition 3.7. *Take* $v, w : Q \to \mathbb{R}$ *with* $v \in L^2(0,T;V)$ *and* $w \in H^1(0,T;V)$*, where* V *is an arbitrary Hilbert space over* Ω*. We denote by* $\overline{v}_\tau$ *the step approximation of* v *and by* $\widehat{\overline{w}}_\tau$*, respectively* $\widehat{w}_\tau$*, the linear approximations with respect to the mean, respectively end, points of* w *on a uniform partition* $\mathcal{P}$*. Then*

$$\|\overline{v}_\tau\|_{L^2(0,T;V)} \le \|v\|_{L^2(0,T;V)} \tag{3.6a}$$

as well as

$$\|\partial_t \widehat{w}_\tau\|_{L^2(0,T;V)} \le \|\partial_t w\|_{L^2(0,T;V)} \quad \textit{and} \quad \left\|\widehat{\overline{w}}_\tau\right\|_{H^1(0,T;V)} \le \left(1 + \frac{1}{\sigma}\right)\|w\|_{H^1(0,T;V)}, \tag{3.6b}$$

where σ *is the uniformity constant of the partition* $\mathcal{P}$ *introduced in the Definition 2.5.*

Proof. For the first one we conclude

$$\|\overline{v}_\tau\|^2_{L^2(0,T;V)} = \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \|\overline{v}^i\|_V^2 \overset{(3.5)}{\leq} \sum_{i=1}^N \tau_i \frac{1}{\tau_i} \int_{t_{i-1}}^{t_i} \|v(\cdot,s)\|_V^2 \,\mathrm{d}s \leq \|v\|^2_{L^2(0,T;V)}.$$

On the other hand we have

$$\begin{aligned}\|\partial_t \widehat{w}_\tau\|^2_{L^2(0,T;V)} &= \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \left\|\frac{w(\cdot,t_i) - w(\cdot,t_{i-1})}{\tau_i}\right\|_V^2 = \sum_{i=1}^N \tau_i \left\|\frac{1}{\tau_i}\int_{t_{i-1}}^{t_i} \partial_t w(\cdot,s)\,\mathrm{d}s\right\|_V^2 \\ &\overset{(3.5)}{\leq} \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \|\partial_t w\|_V^2 = \|\partial_t w\|^2_{L^2(0,T;V)}.\end{aligned}$$

The last estimate we split into the terms $\left\|\widehat{\overline{w}}_\tau\right\|^2_{L^2(0,T;V)}$ and $\left\|\partial_t \widehat{\overline{w}}_\tau\right\|^2_{L^2(0,T;V)}$.

$$\begin{aligned}\left\|\widehat{\overline{w}}_\tau\right\|^2_{L^2(0,T;V)} &= \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \left\|\frac{t-t_{i-1}}{\tau_i}\overline{w}^i + \frac{t_i - t}{\tau_i}\overline{w}^{i-1}\right\|_V^2 \\ &\leq \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \left[2\left(\frac{t-t_{i-1}}{\tau_i}\right)^2 \|\overline{w}^i\|_V^2 + 2\left(\frac{t_i-t}{\tau_i}\right)^2 \|\overline{w}^{i-1}\|_V^2\right] \\ &\leq \sum_{i=1}^N \frac{2}{3}\tau_i \left(\|\overline{w}^i\|_V^2 + \|\overline{w}^{i-1}\|_V^2\right) \\ &\leq \int_{t_0}^{t_1} \|w(\cdot,s)\|_V^2 \,\mathrm{d}s + \sum_{i=2}^N \left(\int_{t_{i-1}}^{t_i} \|w(\cdot,s)\|_V^2 \,\mathrm{d}s + \frac{\tau_i}{\tau_{i-1}} \int_{t_{i-2}}^{t_{i-1}} \|w(\cdot,s)\|_V^2 \,\mathrm{d}s\right) \\ &\leq \left(1 + \frac{1}{\sigma}\right) \|w\|^2_{L^2(0,T;V)},\end{aligned}$$

where we used the uniformity of the partition $\mathcal{P}$, since $\frac{\tau_i}{\tau_{i-1}} \leq \frac{\tau}{\sigma\tau} = \frac{1}{\sigma}$.

Finally we conclude for the other seminorm using the mean value theorem for integration

$$\begin{aligned}\left\|\partial_t \widehat{\overline{w}}_\tau\right\|^2_{L^2(0,T;V)} &= \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \left\|\frac{\overline{w}^i - \overline{w}^{i-1}}{\tau^i}\right\|_V^2 \\ &= \sum_{i=2}^N \frac{1}{\tau_i} \left\|\frac{1}{\tau_i}\int_{t_{i-1}}^{t_i} w(s)\,\mathrm{d}s - \frac{1}{t_{i-1}}\int_{t_{i-2}}^{t_{i-1}} w(s)\,\mathrm{d}s\right\|_V^2 \\ \exists s_i \in (t_{i-1}, t_i) \quad &= \sum_{i=2}^N \frac{1}{\tau^i} \left\|\frac{w(s_i) - w(s_{i-1})}{\tau_i}\right\|_V^2 = \sum_{i=2}^N \frac{1}{\tau_i} \left\|\int_{s_{i-1}}^{s_i} \partial_t w(s)\,\mathrm{d}s\right\|_V^2 \\ &\leq \sum_{i=2}^N \frac{s_i - s_{i-1}}{\tau_i} \int_{s_{i-1}}^{s_i} \|\partial_t w(s)\|_V^2 \,\mathrm{d}s \leq \left(1 + \frac{1}{\sigma}\right) \|\partial_t w\|^2_{L^2(0,T;V)},\end{aligned}$$

where we used in the last step the following estimate

$$\frac{s_i - s_{i-1}}{\tau_i} \leq \frac{t_i - t_{i-2}}{t_i - t_{i-1}} = \frac{t_i - t_{i-1} + t_{i-1} - t_{i-2}}{t_i - t_{i-1}} = 1 + \frac{\tau_{i-1}}{\tau_i} \leq 1 + \frac{1}{\sigma}.$$

□

Lemma 3.8. *Take $z \in L^2(Q)$ and let $\pi : Q \times \mathbb{R} \to \mathbb{R}$ fulfil the assumptions* (2.2e) *and* (2.2j). *Furthermore, we denote by $\overline{\pi}_{\tau_n}$ and $\overline{z}_{\tau_n}$ their step approximations on a family of uniform partitions $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ such that $\tau_n \to 0$ as $n \to \infty$. Then there holds*

$$\overline{\pi}_{\tau_n}(\overline{z}_{\tau_n}) \to \pi(z) \quad \text{in } L^2(Q) \quad \text{as } n \to \infty. \tag{3.7}$$

Proof. Firstly, by the Lebesgue-Besicovitch differentiation theorem we observe

$$\|\overline{z}_{\tau_n}(\cdot,t) - z(\cdot,t)\|_{L^2(\Omega)} \to 0 \quad \text{for a.a. } t \in (0,T) \quad \text{as} \quad n \to \infty.$$

Hence it also holds $\|\overline{z}_{\tau_n}(\cdot,t) - z(\cdot,t)\|_{L^2(Q)} \to 0$ a $n \to \infty$ with the help of Proposition 3.7 and the Lebesgue dominated convergence theorem. Now we use the triangle inequality

$$\|\overline{\pi}_{\tau_n}(\overline{z}_{\tau_n}) - \pi(z)\|^2_{L^2(Q)} \le \|\overline{\pi}_{\tau_n}(\overline{z}_{\tau_n}) - \pi(\overline{z}_{\tau_n})\|^2_{L^2(Q)} + \|\pi(\overline{z}_{\tau_n}) - \pi(z)\|^2_{L^2(Q)}. \tag{3.8}$$

For the first norm we can apply again the Lebesgue-Besicovitch Differentiation theorem and hence

$$\frac{1}{\tau^n_{i_n}} \int_{t^n_{i_n-1}}^{t^n_{i_n}} \left|\pi(x,s,z^{i_n}(x)) - \pi(x,t,z^{i_n}(x))\right|^2 \mathrm{d}s \to 0 \quad \text{for a.a. } (x,t) \in Q \quad \text{as} \quad n \to \infty,$$

where the sequence $\{i_n\}$ is chosen depending on t such that $t \in (t_{i_n-1}, t_{i_n})$. Thus the convergence holds also in $L^2(Q)$. The second norm in (3.8) can be estimated with the help of the uniform Lipschitz continuity assumption on π

$$\|\pi(\overline{z}_{\tau_n}) - \pi(z)\|^2_{L^2(Q)} \le L^2_\pi \|\overline{z}_{\tau_n} - z\|_{L^2(Q)}.$$

□

Lemma 3.9. *Take $z \in L^2(Q)$ and let $\beta : Q \times \mathbb{R} \to \mathbb{R}$ fulfil the almost uniform Lipschitz continuity assumption* (2.2d). *In addition β_{τ_n} and $\overline{z}_{\tau_n}$ denote their step approximations in the end, respectively mean, points on a family of uniform partitions $\{\mathcal{P}_n\}$ such that $\tau_n \to 0$ as $n \to \infty$. Then there holds*

$$\beta_{\tau_n}(\overline{z}_{\tau_n}) \to \beta(z) \quad \text{a.e. in } Q \text{ and weakly in } L^2(Q) \quad \text{as} \quad n \to \infty. \tag{3.9}$$

Proof. To gain the almost everywhere convergence, it suffices to show that for every $\delta \in (0,1)$:

$$\beta_{\tau_n}(\overline{z}_{\tau_n}) \to \beta(z) \quad \text{almost uniformly in } Q^\delta := \{(x,t) \in Q : \delta \le z(x,t) \le 1/\delta\},$$

whenever $\overline{z}_\tau \to z$ a.e. in Q. Thus fix $\delta \in (0,1)$ and take $\eta > 0$. To show is the existence of a subset $Q^\delta_\eta \subset Q^\delta$ such that $\left|Q^\delta \backslash Q^\delta_\eta\right| \le \eta$ and $\beta_{\tau_n}(\overline{z}_{\tau_n}) \to \beta(z)$ uniformly in Q^δ_η, where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^4$. With the help of the Severini-Egorov theorem a set $Q^\delta_\eta \subset Q^\delta$ is found such that $\left|Q^\delta \backslash Q^\delta_\eta\right| \le \eta$ and $\overline{z}_{\tau_n} \to z$ uniformly in Q^δ_η. Now by the assumptions, β is uniform Lipschitz continuous on Q^δ_η with $\operatorname{lip}\beta =: L_{\beta,\delta}$ admitting the estimate

$$\|\beta_{\tau_n}(\overline{z}_{\tau_n}) - \beta(\overline{z}_{\tau_n})\|_{L^2(Q^\delta_\eta)} \le \|\beta_{\tau_n}(\overline{z}_{\tau_n}) - \beta(\overline{z}_{\tau_n})\|_{L^2(Q^\delta_\eta)} \le L_{\beta,\delta}\tau_n.$$

With the triangle inequality we further archieve

$$\begin{aligned}\|\beta_{\tau_n}(\overline{z}_{\tau_n})-\beta(z)\|^2_{L^2(Q^\delta_\eta)} &\leq \|\beta_{\tau_n}(\overline{z}_{\tau_n})-\beta(\overline{z}_{\tau_n})\|^2_{L^2(Q^\delta_\eta)}+\|\beta(\overline{z}_{\tau_n})-\beta(z)\|^2_{L^2(Q^\delta_\eta)}\\ &\leq L^2_{\beta,\delta}(\tau_n)^2+L^2_{\beta,\delta}\,\|\overline{z}_{\tau_n}-z\|^2_{L^2(Q)}\end{aligned}$$

and deduce that $\beta_{\tau_n}(\overline{z}_{\tau_n})$ converges to $\beta(z)$ uniformly on Q^δ_η. Hence the almost everywhere convergence is obtained, which is implied by the almost uniform convergence. With the help of the the last norm estimate and the just proved almost everywhere convergence the weak convergence in $L^2(Q)$ can be concluded with the help of [17, Lemme 1.3, pp. 12-13]. □

3.3 Approximation by extension, regularization and truncation

Let $\beta : Q\times(0,\infty)\to\mathbb{R}$ fulfil the assumptions (2.2d)-(2.2g). Then take a partition $\mathcal{P}$ with diameter τ and define on that the linear approximation $\widehat{\beta}_\tau$ for the endpoints according to Definition 2.7.

3.3.1 Extension

Definition 3.10 (Extension operator)**.** Define the both domains:

$$\Omega_\varepsilon := \{x\in\Omega : \operatorname{dist}(x,\Gamma)<\varepsilon\} \quad\text{and}\quad \Omega'_\varepsilon := \left\{x\in\mathbb{R}^3\backslash\overline{\Omega} : \operatorname{dist}(x,\Gamma)<\varepsilon\right\}.$$

Let $\varepsilon_0\in(0,1)$ be such that for every $x\in\Omega'_{\varepsilon_0}$ exists a unique point $\widetilde{x}\in\Omega_{\varepsilon_0}$ satisfying

$$x' := \frac{x+\widetilde{x}}{2}\in\Gamma \quad\text{and } x-\widetilde{x} \text{ is orthogonal to } \Gamma \text{ at } x'. \tag{3.10a}$$

The correspondence $x\mapsto\widetilde{x}$ is a bi-Lipschitz diffeomorphism of class C^1 (as Γ is of class C^2) from Ω'_{ε_0} onto Ω_{ε_0}.

Now, we set $\widetilde{\Omega} := \overline{\Omega}\cup\Omega'_{\varepsilon_0}$. We define with $v\in L^\infty(\Omega)$ the extension operator

$$\mathcal{E} : L^\infty(\Omega)\to L^\infty(\widetilde{\Omega}), \qquad \mathcal{E}v(x) := \begin{cases} v(x) & \text{if } x\in\Omega\\ v(\widetilde{x}) & \text{if } x\in\Omega'_{\varepsilon_0}\end{cases} \tag{3.10b}$$

Proposition 3.11 (Properties of $\mathcal{E}$)**.** *The extension operator $\mathcal{E}$ is linear and continuous. Moreover one has*

- $\sup\operatorname{ess}_{\widetilde{\Omega}}\mathcal{E}v = \sup\operatorname{ess}_\Omega v$ *and* $\inf\operatorname{ess}_{\widetilde{\Omega}}\mathcal{E}v = \inf\operatorname{ess}_\Omega v$ *for every* $v\in L^\infty(\Omega)$.

And also the following properties apply to all $v\in L^\infty(\Omega)$:

- $\mathcal{E}\geq 0$ *a.e. in* $\widetilde{\Omega}$ *whenever* $v\geq 0$ *a.e. in* Ω
- $\|\nabla\mathcal{E}v\|_{L^\infty(\widetilde{\Omega})}\leq M\,\|\nabla v\|_{L^\infty(\Omega)}$ *if* $\nabla v\in L^\infty(\Omega)$
- $\operatorname{lip}(\mathcal{E}v)\leq M\operatorname{lip}v$ *if v is Lipschitz continuous,*

for some constant M, where $\operatorname{lip} v$ *is the Lipschitz constant of* v*.*

At this point, we define for $\varepsilon \in (0,1)$ the functions $\widetilde{\beta}^i, \widetilde{\beta}^i_\varepsilon : \widetilde{\Omega} \times \mathbb{R} \to \mathbb{R}$ by using the extension operator $\mathcal{E}$

$$\widetilde{\beta}^i(x,r) := \left(\mathcal{E}\beta^i(\cdot,r)\right)(x) \tag{3.12a}$$

$$\widetilde{\beta}^i_\varepsilon(x,r) := \widetilde{\beta}^i(x,r_\varepsilon) \quad \text{for } i = 1,\dots,N, \text{ where } r_\varepsilon := \max\{\varepsilon, \min\{r, 1/\varepsilon\}\}. \tag{3.12b}$$

In all of the following statements on $\widetilde{\beta}^i$, $\widetilde{\beta}^i_\varepsilon$ and later β^i_ε we do not stress out explicitly every time for all $i = 1,\dots,N$.

We observe that $\widetilde{\beta}^i_\varepsilon$ is globally Lipschitz continuous. Indeed, by recalling (2.2d) as well as (3.11) and setting for convenience

$$L_\delta := \operatorname{lip} \beta|_{Q\times[\delta,1/\delta]} \quad \text{for } \delta \in (0,1) \tag{3.13}$$

we conclude $\operatorname{lip} \widetilde{\beta}^i_\varepsilon \le M L_\varepsilon$ by the Definition (3.12b). Moreover, as $\mathcal{E}$ is linear and thanks to (3.11) we infer that both $\widetilde{\beta}(x,t,\cdot)$ and $\widetilde{\beta}_\varepsilon(x,t,\cdot)$ are nondecreasing on $\mathbb{R}$ for every $x \in \widetilde{\Omega}$. Furthermore, both $\widetilde{\beta}^i$ and $\widetilde{\beta}^i_\varepsilon$ vanish at $r = 1$ due to (2.2i). In particular, their values at every $r \in \mathbb{R}$ have the sign of $r - 1$.

3.3.2 Regularization and truncation

Let $\zeta \in C^\infty(\mathbb{R}^4)$ be supported in the unit ball U of $\mathbb{R}^4$ and normalised in $L^1(\mathbb{R}^4)$. Then, by assuming $\varepsilon_0 \le T$ and $\varepsilon \in (0,\varepsilon_0)$ (such restrictions are not stressed in the sequel, but it is understood that they are satisfied), we recall (3.13) and set

$$\delta_\varepsilon := \frac{\varepsilon}{1 + L_\varepsilon} \quad \text{and} \quad \zeta_\varepsilon(x,r) := \delta_\varepsilon^{-4}\zeta\left((x,r)/\delta_\varepsilon\right) \quad \text{for } (x,r) \in \mathbb{R}^4 \tag{3.14a}$$

$$\begin{aligned} \beta^i_\varepsilon(x,r) &:= (\widetilde{\beta}^i_\varepsilon * \zeta_\varepsilon)(x,r) = \int_{\delta_\varepsilon U} \widetilde{\beta}^i_\varepsilon(x-y, r-s)\,\zeta_\varepsilon(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_U \widetilde{\beta}^i_\varepsilon(x - \delta_\varepsilon y, r - \delta_\varepsilon s)\,\zeta(y,s)\,\mathrm{d}y\,\mathrm{d}s \quad \text{for } (x,r) \in \Omega \times \mathbb{R}. \end{aligned} \tag{3.14b}$$

3.3.3 Properties

With the above choice of δ_ε we can estimate and observe

$$\begin{aligned} |\beta^i_\varepsilon(x,r) - \widetilde{\beta}^i_\varepsilon(x,r)| &= \left| \int_U \left(\widetilde{\beta}^i_\varepsilon(x - \delta_\varepsilon y, r - \delta_\varepsilon s) - \widetilde{\beta}^i_\varepsilon(x,r) \right) \zeta(y,s)\,\mathrm{d}y\,\mathrm{d}s \right| \\ &\le \int_U M L_\varepsilon\,|(\delta_\varepsilon y, \delta_\varepsilon s)|\;\zeta(y,s)\,\mathrm{d}y\,\mathrm{d}s \le M L_\varepsilon \delta_\varepsilon \le M\varepsilon \text{for every } (x,r) \in \Omega \times \mathbb{R} \end{aligned} \tag{3.15}$$

since $M L_\varepsilon$ is a Lipschitz constant of $\widetilde{\beta}^i_\varepsilon$, as just observed. Actually, (3.15) holds with the constant M that makes (3.11) true. With a similar argument, we see that

$$\beta^i_\varepsilon \text{ is Lipschitz continuous with} \quad \operatorname{lip} \beta^i_\varepsilon \le M L_\varepsilon \tag{3.16}$$

since such a property holds for $\widetilde{\beta}^i_\varepsilon$. In the sequel we use the following more precise facts

$$\sup_{\Omega\times[\delta,1/\delta]} |\beta^i_\varepsilon| \le \sup_{\widetilde{\Omega}\times[\delta/2,1/\delta+\delta/2]} |\widetilde{\beta}^i| \le \sup_{\Omega\times[\delta/2,1/\delta+\delta/2]} |\beta(\cdot,t_i,\cdot)| \tag{3.17a}$$

$$\operatorname{lip} \beta^i_\varepsilon|_{\Omega\times[\delta,1/\delta]} \le \operatorname{lip} \widetilde{\beta}^i|_{\widetilde{\Omega}\times[\delta/2,1/\delta+\delta/2]} \le \operatorname{lip} \beta|_{\Omega\times(t_{i-1},t_i]\times[\delta/2,1/\delta+\delta/2]}, \tag{3.17b}$$

for $\delta \in (0,1)$ and $\varepsilon \le \delta/2$. Indeed, we have $\delta_\varepsilon \le \varepsilon \le \delta/2$. Hence, if $(x,t) \in Q$ and $\delta \le r \le 1/\delta$, the values of $\widetilde{\beta}^i_\varepsilon$ in (3.14b) actually are values of $\widetilde{\beta}^i$ at points of the set $\widetilde{\Omega} \times [\delta/2, 1/\delta + \delta/2]$, where $\widetilde{\beta}^i$ is bounded and Lipschitz continuous. Therefore, both the supremum (3.17a) and the Lipschitz constant (3.17b) are preserved by the convolution since ζ is normalised. Finally, we point out that

$$\beta^i_\varepsilon(x,\cdot) \text{ is nondecreasing on } \mathbb{R} \text{ for every } x \in \Omega \tag{3.18}$$

since such a property holds for $\widetilde{\beta}^i_\varepsilon$ and ζ is nonnegative.

Proposition 3.12. *We have*

$$|\beta^i_{\varepsilon,x}(x,r)| \le c(1 + |\beta^i_\varepsilon(x,r)|) \tag{3.19}$$

for every $(x,r) \in \Omega \times \mathbb{R}$*, some constant* c *and* ε *small enough.*

Proof. Let us refer to [2, Proposition 3.3]. □

Lemma 3.13. *Assume* $z, z_n \in L^2(\Omega), z > 0$ *a.e. in* Ω*, and* $z_n \to z$ *a.e. in* Ω*. Moreover, let* $\{\varepsilon_n\}$ *be a positive real sequence converging to* 0*. Then,* $\{\beta^i_{\varepsilon_n}(z_n)\}$ *converges to* $\beta^i(z)$ *a.e. in* Ω*.*

Proof. It suffices to show that for every $\delta \in (0,1)$ we have

$$\beta^i_{\varepsilon_n}(z_n) \to \beta^i(z) \quad \text{almost uniformly in } \Omega^\delta := \{x \in \Omega : \delta \le z(x) \le 1/\delta\}.$$

For applying the Severini-Egorov theorem we fix $\delta \in (0,1)$ and $\eta > 0$. Therewith, we find $\Omega^\delta_\eta \subset \Omega^\delta$ such that $\left|\Omega \backslash \Omega^\delta_\eta\right| \le \eta$ and $z_n \to z$ uniformly in Ω^δ_η. Now fix $\overline{n}$ such that

$$\varepsilon_n \le \frac{\delta}{2} \quad \text{and} \quad \frac{\delta}{2} \le z_n \le \frac{2}{\delta} \quad \text{in } \Omega^\delta_\eta \text{ for every } n \ge \overline{n}.$$

On the other hand, we have

$$\left\|\beta^i_{\varepsilon_n}(z_n) - \beta^i(z)\right\|_{L^\infty(\Omega^\delta_\eta)} \le \left\|\beta^i_{\varepsilon_n}(z_n) - \beta^i_{\varepsilon_n}(z)\right\|_{L^\infty(\Omega^\delta_\eta)} + \left\|\beta^i_{\varepsilon_n}(z) - \beta^i(z)\right\|_{L^\infty(\Omega^\delta_\eta)}.$$

Assume now $n \ge \overline{n}$. Then, $\varepsilon_n \le \delta$, whence $\varepsilon_n \le z \le 1/\varepsilon_n$. Thus, $\beta^i(z) = \widetilde{\beta}^i_{\varepsilon_n}(z)$ by the truncation procedure (3.12). We infer that the last term is bounded by $M\varepsilon_n$ as a consequence of (3.16). On the other hand, as $\varepsilon_n \le \delta/2$. Further as $\varepsilon_n \le \delta/2$ we can use the bound of the Lipschitz constants in (3.17b). Finally, we conclude that

$$\left\|\beta^i_{\varepsilon_n}(z_n) - \beta^i(z)\right\|_{L^\infty(\Omega^\delta_\eta)} \le c_\delta \left\|z_n - z\right\|_{L^\infty(\Omega^\delta_\eta)} + M\varepsilon_n$$

and deduce that $\beta^i_{\varepsilon_n}(z_n)$ converges to $\beta^i(z)$ uniformly in Ω^δ_η. □

3.4 Harmonic extension

Definition 3.14 (Harmonic extension of $\overline{\vartheta}_{\Gamma,\tau}$)**.** Let $\mathcal{P}$ be a uniform partition with diameter τ. Then the vector $\vartheta^{\mathcal{P}} = (\vartheta^1_{\mathcal{H}}, \dots, \vartheta^N_{\mathcal{H}})$ is defined by

$$\vartheta^i_{\mathcal{H}} \in H^1(\Omega), \quad \vartheta^i_{\mathcal{H}}|_\Gamma = \overline{\vartheta}^i_\Gamma, \quad \Delta\vartheta^i_{\mathcal{H}} = 0 \quad \text{in } \Omega, \quad \text{for } i = 1 \dots N. \tag{3.20}$$

The step $\vartheta_{\mathcal{H},\tau}$ respectively linear $\widehat{\vartheta}_{\mathcal{H},\tau}$ interpolation of the vector $\vartheta^{\mathcal{P}}$ are the harmonic extensions of the time approximated boundary values $\overline{\vartheta}_{\Gamma,\tau}$ respectively $\widehat{\overline{\vartheta}}_{\Gamma,\tau}$.

Proposition 3.15. *Let ϑ_Γ fulfil the assumption* (2.6a)*. Then one has for the harmonic extension $\vartheta_{\mathcal{H},\tau} \in L^2(0,T;H^1(\Omega))$ with the estimate*

$$\|\vartheta_{\mathcal{H},\tau}\|_{L^2(0,T;H^1(\Omega))} \le c\,\|\vartheta_\Gamma\|_{L^2(0,T:H^{1/2}(\Gamma))} \quad \text{and} \quad \vartheta_* \le \vartheta_{\mathcal{H},\tau} \le \vartheta^* \quad \text{a.e. in } Q. \tag{3.21a}$$

as well as $\widehat{\vartheta}_{\mathcal{H},\tau} \in H^1(0,T;L^2(\Omega))$ bounded by

$$\left\|\widehat{\vartheta}_{\mathcal{H},\tau}\right\|_{H^1(0,T;L^2(\Omega))} \le c\,\|\vartheta_\Gamma\|_{H^1(0,T;H^{-1/2}(\Gamma))} \quad \text{and} \quad \vartheta_* \le \widehat{\vartheta}_{\mathcal{H},\tau} \le \vartheta^* \quad \text{a.e. in } Q. \tag{3.21b}$$

Proof. Due to the maximum principle for elliptic equations we find that $\vartheta_* \le \vartheta^i_{\mathcal{H}} \le \vartheta^*$ a.e. in Ω for all $i = 1, \dots, N$. Thus the same bounds are valid for $\vartheta_{\mathcal{H},\tau}$ and $\widehat{\vartheta}_{\mathcal{H},\tau}$. Further it holds $\|\vartheta^i_{\mathcal{H}}\|_{H^1(\Omega)} \le \left\|\overline{\vartheta}^i_\Gamma\right\|_{H^{1/2}(\Omega)}$, due to the general theory of elliptic equations. With the definition of the harmonic extension it yields

$$\|\vartheta_{\mathcal{H},\tau}\|^2_{L^2(0,T;H^1(\Omega))} = \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \|\vartheta^i_{\mathcal{H}}\|^2_{H^1(\Omega)} \le \|\overline{\vartheta}_{\Gamma,\tau}\|^2_{L^2(0,T;H^{1/2}(\Gamma))} \le \|\vartheta_\Gamma\|^2_{L^2(0,T;H^{1/2}(\Gamma))},$$

where the last estimate is a consequence of (3.6a). Also from the theory of elliptic equations we can conclude $\|\vartheta^i_{\mathcal{H}}\|_{L^2(\Omega)} \le \left\|\overline{\vartheta}^i_\Gamma\right\|_{H^{-1/2}(\Omega)}$. Now due to the linearity of the Laplace equation (3.20) we can estimate

$$\begin{aligned}
\left\|\widehat{\vartheta}_{\mathcal{H},\tau}\right\|^2_{L^2(0,T;L^2(\Omega))} &= \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \left\|\frac{t - t_{i-1}}{\tau_i}\vartheta^i_{\mathcal{H}} + \frac{t_i - t}{\tau_i}\vartheta^{i-1}_{\mathcal{H}}\right\|^2_{L^2(\Omega)} \\
&\le \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \left\|\frac{t - t_{i-1}}{\tau_i}\overline{\vartheta}^i_\Gamma + \frac{t_i - t}{\tau_i}\overline{\vartheta}^{i-1}_\Gamma\right\|^2_{H^{-1/2}(\Gamma)} \\
&\le \left\|\widehat{\overline{\vartheta}}_{\Gamma,\tau}\right\|^2_{L^2(0,T;H^{-1/2}(\Gamma))} \le \left(1 + \frac{1}{\sigma}\right)\|\vartheta_\Gamma\|^2_{L^2(0,T;H^{-1/2}(\Gamma))},
\end{aligned}$$

where we have used Propoistion 3.7 in the last inequality.

Finally we show the bound in the H^1-seminorm of $\widehat{\vartheta}_{\mathcal{H},\tau}$

$$\begin{aligned}
\left\|\partial_t\widehat{\vartheta}_{\mathcal{H},\tau}\right\|^2_{L^2(0,T;L^2(\Omega))} &= \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \left\|\frac{\vartheta^i_{\mathcal{H}} - \vartheta^{i-1}_{\mathcal{H}}}{\tau_i}\right\|^2_{L^2(\Omega)} \le \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \left\|\frac{\overline{\vartheta}^i_\Gamma - \overline{\vartheta}^{i-1}_\Gamma}{\tau_i}\right\|^2_{H^{-1/2}(\Gamma)} \\
&= \left\|\partial_t\widehat{\overline{\vartheta}}_{\Gamma,\tau}\right\|^2_{L^2(0,T;H^{-1/2}(\Gamma))} \le \left(1 + \frac{1}{\sigma}\right)\|\partial_t\vartheta_\Gamma\|^2_{L^2(0,T;H^{-1/2}(\Gamma))}.
\end{aligned}$$

where again the last estimate is a conclusion from Proposition 3.7. ∎

Proposition 3.16. *We have for a constant $c > 0$*

$$\left\|\operatorname{Ln}_\varepsilon \vartheta^i_{\mathcal{H}}\right\|_{L^\infty(\Omega)\cap H^1(\Omega)} \leq c \quad \text{and} \quad \left\|\beta^i_\varepsilon(\vartheta^i_{\mathcal{H}})\right\|_{L^\infty(\Omega)\cap H^1(\Omega)} \leq c, \tag{3.22a}$$

for all $1 \leq i \leq N$ and ε small enough as well as for a partition $\mathcal{P}$ with $\tau > 0$

$$\|\beta_\tau(\vartheta_{\mathcal{H},\tau})\|_{L^\infty(Q)} \leq c \quad \text{and} \quad \left\|\partial_t \widehat{\beta}_\tau(\vartheta_{\mathcal{H},\tau})\right\|_{L^2(Q)} \leq c. \tag{3.22b}$$

Proof. The estimates (3.21a) and (3.3d) of Proposition 3.3 imply that $\ell_* \leq \ln_\varepsilon \vartheta^i_{\mathcal{H}} \leq \ell^*$ a.e. in Ω. Thus also $\operatorname{Ln}_\varepsilon \vartheta^i_{\mathcal{H}}$ is bounded. For the H^1-norm we argue in the same way, but now using the estimates (3.21a) and (3.3d).

The L^∞ bound of β^i_ε is again just a consequence of (3.21a), (2.3) and (3.17a). For the estimate regarding the space derivative we have

$$\begin{aligned}\left\|\nabla \beta^i_\varepsilon(\vartheta^i_{\mathcal{H}})\right\|_{L^2(\Omega)} &\leq \left\|\beta^i_{\varepsilon,x}(\vartheta^i_{\mathcal{H}})\right\|_{L^2(\Omega)} + \left\|(\beta^i_\varepsilon)'(\vartheta^i_{\mathcal{H}})\nabla\vartheta^i_{\mathcal{H}}\right\|_{L^2(\Omega)} \\ &\leq c\left\|\beta^i_{\varepsilon,x}(\vartheta^i_{\mathcal{H}})\right\|_{L^\infty(\Omega)} + c \sup_{\Omega\times[\vartheta_*,\vartheta^*]} \left|(\beta^i_\varepsilon)'\right| \left\|\vartheta^i_{\mathcal{H}}\right\|_{H^1(\Omega)} \leq c,\end{aligned}$$

where we used the boundedness of $\left\|\beta^i_{\varepsilon,t}(\vartheta_{\mathcal{H}})\right\|_{L^\infty(\Omega)}$, provided ε is small enough due to Proposition 3.12 and the L^∞–estimate just proved as well as the supremum estimate (3.17a) and finally (3.21a).

The next estimate is just again a consequence of (3.21a), and (2.3)

$$\begin{aligned}\left\|\partial_t \widehat{\beta}_\tau(\vartheta_{\mathcal{H},\tau})\right\|^2_{L^2(Q)} &\leq \sum_{i=1}^N \tau_i \left\|\frac{\beta^i(\vartheta^i_{\mathcal{H}}) - \beta^i(\vartheta^{i-1}_{\mathcal{H}})}{\tau^i}\right\|^2_{L^2(\Omega)} \\ &\leq \sum_{i=1}^N \frac{1}{\tau_i}\left(\left\|\beta^i(\vartheta^i_{\mathcal{H}}) - \beta^{i-1}(\vartheta^i_{\mathcal{H}})\right\|^2_{L^2(\Omega)} + \left\|\beta^{i-1}(\vartheta^i_{\mathcal{H}}) - \beta^i(\vartheta^{i-1}_{\mathcal{H}})\right\|^2_{L^2(\Omega)}\right) \\ &\leq \sum_{i=1}^N \frac{1}{\tau_i}\left(L_{\beta,\vartheta_*}\tau_i^2 + L^2_{\beta,\vartheta_*}\left\|\vartheta^i_{\mathcal{H}} - \vartheta^{i-1}_{\mathcal{H}}\right\|_{L^2(Q)}\right) \leq c + c\left\|\partial_t\widehat{\vartheta}_{\mathcal{H}}\right\|_{L^2(Q)} \leq c,\end{aligned}$$

where L_{β,ϑ_*} is the Lipschitz constant of β on the interval $[\vartheta_*, \vartheta^*]$, due to the boundedness of $\left\|\partial_t\widehat{\vartheta}_{\mathcal{H}}\right\|_{L^2(Q)}$ pointed out in (3.21b). □

4 Time-step discretisation

In this section we prove the existence and uniqueness result as well as the maximum principle for χ. The discrete system consists of the following equations

$$\frac{\vartheta^i - \vartheta^{i-1}}{\tau_i} - G'(\chi^i)\frac{\chi^i - \chi^{i-1}}{\tau_i} - \Delta \ln\vartheta^i + \xi^i = \overline{\pi}^i(\vartheta^{i-1}), \quad \xi^i = \beta^i(\vartheta^i) \tag{4.1a}$$

$$\frac{\chi^i - \chi^{i-1}}{\tau_i} - \Delta\chi^i + F'(\chi^i) + G'(\chi^i)\vartheta^{i-1} = 0 \tag{4.1b}$$

$$\ln\vartheta^i = \ln\overline{\vartheta}^i_\Gamma \quad \text{and} \quad \partial_n\chi^i = 0 \quad \text{on } \Gamma, \quad \text{for } i = 1, \ldots, N \tag{4.1c}$$

$$\vartheta^0 = \vartheta_0 \quad \text{and} \quad \chi^0 = \chi_0 \quad \text{in } \Omega \tag{4.1d}$$

where the initial and boundary values are bounded due to the assumptions (2.6)

$$\vartheta_* \le \overline{\vartheta}_\Gamma^i \le \vartheta^* \quad \text{on } \Gamma, \tag{4.1e}$$

$$\vartheta_* \le \vartheta_0 \le \vartheta^* \quad \text{and} \quad \chi_* \le \chi_0 \le \chi^* \quad \text{in } \Omega \quad \text{for } i = 1, \dots, N. \tag{4.1f}$$

Remark 4.1. (i) We point out that the equation for ϑ^i (4.1a) is in semi-implicit form and the equation for χ^i (4.1b) is separated from the first equation (4.1a) on level i.

(ii) We observe: if there exists a solution of (4.1a) then this solution satisfies $\vartheta^i > 0$ a.e. in Ω.

At this point we assume that we have solved the discrete problem up to the time t_{i-1}. In addition χ^{i-1} is bounded: $\chi_* \le \chi^{i-1} \le \chi^*$. Then, for getting to time t_i, three steps are needed.

1) We obtain χ^i by solving (4.1b).

2) We solve a regularised version of (4.1a) resulting in some ϑ^i_ε.

3) We perform the limit $\varepsilon \to 0$ in the approximation to get the exact solution ϑ^i of (4.1a).

4.1 Solving for χ^i

4.1.1 Maximum principle

Let us fix a Lipschitz continuous function $H : \mathbb{R} \to \mathbb{R}$ of class C^1 such that

$$H(r) = 0 \quad \text{if } r \in [\chi_*, \chi^*] \quad \text{and} \quad H'(r) > 0 \quad \text{if } r \notin [\chi_*, \chi^*]. \tag{4.2a}$$

Now, we use $H(\chi)$ as a test function for (4.1b) and integrate over Ω. After integrating by parts, we obtain

$$\int_\Omega \frac{\chi^i - \chi^{i-1}}{\tau_i} H(\chi^i) + \int_\Omega \left|\nabla \chi^i\right|^2 H'(\chi^i) + \int_\Omega F'(\chi^i) H(\chi^i) + \int_\Omega G'(\chi')\vartheta^{i-1} H(\chi^i) = 0 \tag{4.2b}$$

The second integral is obviously nonnegative. Also the third and the fourth are nonnegative because F' respectively G' and H have the same sign due to (2.2c) and (4.2a) as well as ϑ^{i-1} is nonnegative. For the first integral we have $\chi^{i-1} \in [\chi_*, \chi^*]$ by assumption. Adding the zero $-\frac{\chi^i - \chi^{i-1}}{\tau_i} H(\chi^{i-1})$ leads to

$$\int_\Omega \frac{\chi^i - \chi^{i-1}}{\tau_i} H(\chi^i) = \int_\Omega \frac{\chi^i - \chi^{i-1}}{\tau_i} (H(\chi^i) - H(\chi^{i-1})) \ge 0,$$

because H is monotone. We have deduced that also the first integral is nonnegative. Finally, all integrals are identically zero and by Definition (4.2a) of H we obtain

$$\chi_* \le \chi^i \le \chi^* \quad \text{a.e. in } \Omega. \tag{4.2c}$$

4.1.2 Existence

Let $\Phi : L^2(\Omega) \to L^2(\Omega)$ be the map which solves for a given function $\widetilde{\chi}^i \in L^2(\Omega)$ the system

$$\frac{\Phi(\widetilde{\chi}^i)}{\tau_i} - \Delta\Phi(\widetilde{\chi}^i) = \frac{\chi^{i-1}}{\tau_i} - F'(\widetilde{\chi}^i) - G'(\widetilde{\chi}^i)\vartheta^{i-1} \quad \text{in } \Omega \tag{4.3a}$$

$$\partial_n \chi^i = 0 \quad \text{on } \Gamma. \tag{4.3b}$$

Then, from the theory of elliptic equations we have a unique solution satisfying

$$\left\|\Phi(\widetilde{\chi}^i)\right\|_{H^2_n(\Omega)} \le C\left(\frac{1}{\tau_i}\left\|\chi^{i-1}\right\|_{L^2(\Omega)} + L_F + L_G\left\|\vartheta^{i-1}\right\|_{L^2(\Omega)}\right) \le C, \tag{4.3c}$$

due to the regularity of Γ and the boundedness of F' and G' remarked in (2.8). In addition, C is independent of $\widetilde{\chi}^i$. Thus, the range of Φ is a compact convex subset of $L^2(\Omega)$.

Let $\left\{\widetilde{\chi}^i_n\right\}_{n\in\mathbb{N}}$ be a sequence such that $\widetilde{\chi}^i_n \to \widetilde{\chi}^i$ in $L^2(\Omega)$. Then also $\Phi(\widetilde{\chi}^i_n) \in H^2_n(\Omega)$ and at least there exists a subsequence $\{n_k\}_{k\in\mathbb{N}}$ with $\widetilde{\chi}^i_{n_k} \to \widetilde{\chi}^i$ a.e. in Ω. As a consequence for this subsequence $\{n_k\}$, we also get the convergence of the right-hand side a.e. in Ω and also in $L^2(\Omega)$ by Lebesgue dominated convergence theorem. From this it follows that $\Phi(\widetilde{\chi}^i_{n_k}) \to \Phi(\widetilde{\chi}^i)$ in $H^2(\Omega)$. The difference between this limit and another subsequence with different limit would solve a homogeneous Poisson equation and hence would coincide. Thus we obtain

$$\Phi(\widetilde{\chi}^i_n) \to \Phi(\widetilde{\chi}^i) \quad \text{in } H^2(\Omega). \tag{4.3d}$$

Therefore, Φ is also a continuous operator. The Schauder fixed point theorem gives the existence of at least one $\chi^i \in H^2_n(\Omega)$ with $\chi^i = \Phi(\chi^i)$, hence a solution of (4.1b).

4.1.3 Uniqueness

Let χ^i_1 and χ^i_2 be two fixed points of the above defined map Φ. Testing the difference of the fixed point equations (4.3a) for χ^i_1 and χ^i_2 by $\chi^i_1 - \chi^i_2$ yields

$$\frac{1}{\tau_i}\left\|\chi^i_1 - \chi^i_2\right\|^2_{L^2(\Omega)} + \left\|\nabla\left(\chi^i_1 - \chi^i_2\right)\right\|^2_{L^2(\Omega)} \le L_F\left\|\chi^i_1 - \chi^i_2\right\|^2_{L^2(\Omega)} + L_G\int_\Omega \left|\vartheta^{i-1}\right|\left|\chi^i_1 - \chi^i_2\right|^2,$$

where we have integrated by parts and used the Lipschitz continuity of F and G. The generalised Hölder inequality with exponents $p_1 = 2$ and $p_2 = p_3 = 4$ admits the estimate for the last integral

$$\int_\omega \left|\vartheta^{i-1}\right|\left|\chi^i_1 - \chi^i_2\right|^2 \le \left\|\vartheta^{i-1}\right\|_{L^2(\Omega)}\left\|\chi^i_1 - \chi^i_2\right\|^2_{L^4(\Omega)}.$$

Remind that $H^1(\Omega) \subset L^4(\Omega) \subset L^2(\Omega)$, where the first embedding is compact due to the Sobolev embedding theorem. Then, the norm in $L^4(\Omega)$ can be controlled with [17, Lemme 5.1, p. 58]: for all $\eta > 0$ exists a constant c_η such that

$$\left\|\chi^i_1 - \chi^i_2\right\|^2_{L^4(\Omega)} \le \eta\left\|\nabla\left(\chi^i_1 - \chi^i_2\right)\right\|^2_{L^2(\Omega)} + c_\eta\left\|\chi^i_1 - \chi^i_2\right\|^2_{L^2(\Omega)}, \tag{4.4a}$$

where we have used the seminorm with a factor η instead of the norm in $H^1(\Omega)$. Setting $C := L_G \left\|\vartheta^{i-1}\right\|_{L^2(\Omega)}$ and combining these estimates hold

$$\left(\frac{1}{\tau_i} - L_F - c_\eta C\right) \left\|\chi_1^i - \chi_2^i\right\|^2_{L^2(\Omega)} + (1-\eta C) \left\|\nabla\left(\chi_1^i - \chi_2^i\right)\right\|^2_{L^2(\Omega)} \leq 0 \tag{4.4b}$$

Now, by choosing $\eta < \frac{1}{C}$ and $\tau_i < \frac{1}{L_F + c_\eta C}$ uniqueness is proven.

4.2 Solving for ϑ^i

4.2.1 Uniqueness

We test the difference of the equations (4.1a) for two solutions ϑ_1^i and ϑ_2^i by $\ln\vartheta_1^i - \ln\vartheta_2^i$ and obtain by partial integration

$$\int_\Omega \frac{\vartheta_1^i - \vartheta_2^i}{\tau_i}\left(\ln\vartheta_1^i - \ln\vartheta_2^i\right) + \int_\Omega \left|\nabla\left(\ln\vartheta_1^i - \ln\vartheta_2^i\right)\right|^2 + \int_\Omega \left(\beta^i(\vartheta_1^i) - \beta^i(\vartheta_2^i)\right)\left(\ln\vartheta_1^i - \ln\vartheta_2^i\right) = 0.$$

Obviously, the second integral is nonnegative and also the the third one by monotonicity, hence $\vartheta_1^i = \vartheta_2^i$ a.e. as the logarithm is strictly monotone.

4.2.2 Regularised problem

We substitute β^i with the approximation β_ε^i introduced in (3.14b) and the ln by the regularisation Ln_ε introduced in Definition 3.1.

$$\frac{\vartheta_\varepsilon^i}{\tau_i} - \Delta \mathrm{Ln}_\varepsilon \vartheta_\varepsilon^i + \beta_\varepsilon^i(\vartheta_\varepsilon^i) = h^i \tag{4.5a}$$

$$\vartheta_\varepsilon^i = \overline{\vartheta}_\Gamma^i \quad \text{on } \Gamma, \quad \text{for } i = 1, \ldots, N \tag{4.5b}$$

$$\vartheta^0 = \vartheta_0 \quad \text{in } \Omega, \tag{4.5c}$$

where h^i is defined by

$$h^i := \frac{\vartheta^{i-1}}{\tau_i} + G'(\chi^i)\frac{\chi^i - \chi^{i-1}}{\tau_i} + \overline{\pi}^i(\vartheta^{i-1}). \tag{4.5d}$$

By assumptions on G, π and the already stated solution χ^i we infer that $h^i \in L^2(\Omega)$.

4.2.3 Existence and uniqueness for $\varepsilon > 0$

We set $u_\varepsilon^i := \mathrm{Ln}_\varepsilon \vartheta_\varepsilon^i - u_\mathcal{H}^i$, where $u_\mathcal{H}^i$ is the harmonic extension of $\mathrm{Ln}_\varepsilon \vartheta_\Gamma^i$ into Ω. By the maximum principle for harmonic functions and the assumptions (2.6a) on ϑ_Γ follows

$$\mathrm{Ln}_\varepsilon \vartheta_* \leq u_\mathcal{H}^i \leq \mathrm{Ln}_\varepsilon \vartheta^* \quad \text{a.e. in } \Omega. \tag{4.6}$$

Then the associated problem for u_ε^i is

$$\frac{\mathrm{Ln}_\varepsilon^{-1}\left(u_\varepsilon^i + u_\mathcal{H}^i\right)}{\tau_i} - \Delta u_\varepsilon^i + \beta_\varepsilon^i\left(\mathrm{Ln}_\varepsilon^{-1}\left(u_\varepsilon^i + u_\mathcal{H}^i\right)\right) = h^i \qquad \text{in } \Omega \tag{4.7a}$$

$$u_\varepsilon^i = 0 \qquad \text{on } \Gamma \tag{4.7b}$$

Let $M : L^2(\Omega) \to L^2(\Omega)$ be the operator defined by

$$M(\cdot) := \frac{1}{\tau_i} \operatorname{Ln}_\varepsilon^{-1}\left(\cdot + u_{\mathcal{H}}^i\right) + \beta_\varepsilon^i \left(\operatorname{Ln}_\varepsilon^{-1}\left(\cdot + u_{\mathcal{H}}^i\right)\right).$$

First, M is monotone because $\operatorname{Ln}_\varepsilon^{-1}$ and β_ε are also monotone. In addition, M is hemicontinuous since $\operatorname{Ln}_\varepsilon^{-1}$ and β_ε^i are continuous.

Further, let $L : L^2(\Omega) \to L^2(\Omega)$ be the negative Laplacian $L = -\Delta$. Then, L is a maximal monotone operator with domain $\mathcal{D}(L) = H_0^1(\Omega) \cap H^2(\Omega) \subset L^2(\Omega)$.

Because of the monotonicity of M it follows that $(M(u) - M(\mathbf{0}), u) \geq 0$. Now, take $u \in \mathcal{D}(L)$, then

$$(Lu, u) + (M(u), u) = (\nabla u, \nabla u) + (M(u) - M(\mathbf{0}), u) + (M(\mathbf{0}), u) \geq \|\nabla u\|_{L^2(\Omega)}^2 - C \|u\|_{L^2(\Omega)},$$

where $\mathbf{0} \in L^2(\Omega)$ is the map identical 0 and C is a bound for

$$\|M(\mathbf{0})\|_{L^2(\Omega)} \leq \frac{1}{\tau_i} \left\|\operatorname{Ln}_\varepsilon^{-1}(u_{\mathcal{H}}^i)\right\|_{L^2(\Omega)} + \left\|\beta_\varepsilon^i(\operatorname{Ln}_\varepsilon^{-1}(u_{\mathcal{H}}^i))\right\|_{L^2(\Omega)} =: C.$$

Note that C is bounded due to (4.6), property (3.3d) and Proposition 3.16. We conclude with the Poincaré inequality, where we take a series $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{D}(L)$ diverging $\lim_{n\to\infty} \|u_n\|_{L^2(\Omega)} = \infty$, then

$$\lim_{n\to\infty} \frac{(Lu_n, u_n) + (M(u_n), u_n)}{\|u_n\|_{L^2(\Omega)}} \geq \lim_{n\to\infty} \frac{\|\nabla u_n\|_{L^2(\Omega)}^2}{M_\Omega \|\nabla u_n\|_{L^2(\Omega)}} - C \geq \lim_{n\to\infty} \frac{\|u_n\|_{L^2(\Omega)}}{M_\Omega^2} - C = +\infty,$$

hence $L + M$ is coercive.

Now, by applying [1, Corollary 1.3, p. 48] we conclude that $L + M$ is a maximal monotone operator with

$$\mathcal{R}(L + M) = L^2(\Omega).$$

Consequently, there exists a solution $u_\varepsilon^i \in H_0^1(\Omega)$ for (4.7a). But this leads to a solution $\vartheta_\varepsilon^i \in H^1(\Omega)$, as $u_{\mathcal{H}}^i \in H^2(\Omega)$ and $\operatorname{Ln}_\varepsilon^{-1}(u_\varepsilon^i) \in H_0^1(\Omega)$ for fixed $\varepsilon > 0$.

Uniqueness follows in the same way as already shown in Section 4.2.1 because $\operatorname{Ln}_\varepsilon$ is a strict monotone function by property (3.3c).

4.2.4 A priori estimates

In the following calculations, c is a general constant depending on τ^i and on several norms of $\vartheta_{\mathcal{H}}^i$ listed in Definition 3.14 and Proposition 3.16 respectively.

First a priori estimate We test the equation (4.5a) by $\vartheta_\varepsilon^i - \vartheta_{\mathcal{H}}^i + (\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_{\mathcal{H}}^i)$. The harmonic extension $\vartheta_{\mathcal{H}}^i$ allows us to integrate by parts without a boundary integral, because $\vartheta_\varepsilon^i - \vartheta_\mathcal{H}^i$ fulfils homogeneous Dirichlet boundary conditions.

$$\begin{aligned}
&\frac{1}{\tau_i}\int_\Omega \left(\vartheta_\varepsilon^i - \vartheta_\mathcal{H}^i\right)^2 + \frac{1}{\tau_i}\int_\Omega \left(\vartheta_\varepsilon^i - \vartheta_\mathcal{H}^i\right)\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right) + \int_\Omega \nabla\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i\right)\cdot\nabla\vartheta_\varepsilon^i \\
&\quad + \int_\Omega \left|\nabla\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right)\right|^2 + \int_\Omega \left(\beta_\varepsilon^i(\vartheta_\varepsilon^i) - \beta_\varepsilon^i(\vartheta_\mathcal{H}^i)\right)\left(\left(\vartheta_\varepsilon^i - \vartheta_\mathcal{H}^i\right) + \left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right)\right) \\
&= -\frac{1}{\tau_i}\int_\Omega \vartheta_\mathcal{H}^i\left(\vartheta_\varepsilon^i - \vartheta_\mathcal{H}^i\right) - \frac{1}{\tau_i}\int_\Omega \vartheta_\mathcal{H}^i\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right) + \int_\Omega \nabla\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i\right)\cdot\nabla\vartheta_\mathcal{H}^i \qquad (4.8)\\
&\quad - \int_\Omega \nabla\left(\operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right)\cdot\nabla\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right) \\
&\quad + \int_\Omega \left(h^i - \beta_\varepsilon^i(\vartheta_\mathcal{H}^i)\right)\left(\left(\vartheta_\varepsilon^i - \vartheta_\mathcal{H}^i\right) + \left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right)\right).
\end{aligned}$$

All integrals on the left-hand side are nonnegative due to monotonicity of $\operatorname{Ln}_\varepsilon$ and β_ε. Now, let us estimate the right-hand side. With the help of Young's inequality the first integral becomes

$$-\frac{1}{\tau_i}\int_\Omega \vartheta_\mathcal{H}^i\left(\vartheta_\varepsilon^i - \vartheta_\mathcal{H}^i\right) \le \frac{1}{\tau_i}\int_\Omega \left|\vartheta_\mathcal{H}^i\right|^2 + \frac{1}{4\tau_i}\int_\Omega \left(\vartheta_\varepsilon^i - \vartheta_\mathcal{H}^i\right)^2.$$

For the second one we additionally use Poincaré's inequality

$$\begin{aligned}
-\frac{1}{\tau_i}\int_\Omega \vartheta_\mathcal{H}^i\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right) &\le \frac{2M_\Omega^2}{(\tau_i)^2}\int_\Omega \left|\vartheta_\mathcal{H}^i\right|^2 + \frac{1}{8M_\Omega^2}\int_\Omega \left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right)^2 \\
&\le c\int_\Omega \left|\vartheta_\mathcal{H}^i\right|^2 + \frac{1}{8}\int_\Omega \left|\nabla\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right)\right|^2.
\end{aligned}$$

For the third and fourth we insert proper constants

$$\begin{aligned}
\int_\Omega \nabla\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i\right)\cdot\nabla\vartheta_\mathcal{H}^i &= \int_\Omega \nabla\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right)\cdot\nabla\vartheta_\mathcal{H}^i + \int_\Omega \nabla\left(\operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right)\cdot\nabla\vartheta_\mathcal{H}^i \\
&\le \frac{1}{8}\int_\Omega \left|\nabla\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right)\right|^2 + c\int_\Omega \left|\nabla\vartheta_\mathcal{H}^i\right|^2 + c\int_\Omega \left|\nabla \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right|^2.
\end{aligned}$$

Similarly, it holds

$$-\int_\Omega \nabla \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\cdot\nabla\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right) \le \frac{1}{8}\int_\Omega \left|\nabla\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right)\right|^2 + c\int \left|\nabla \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right|^2.$$

Finally, we conclude with the last integral, again using Poincaré's inequality

$$\begin{aligned}
&\int_\Omega \left(h^i - \beta_\varepsilon^i(\vartheta_\mathcal{H}^i)\right)\left(\left(\vartheta_\varepsilon^i - \vartheta_\mathcal{H}^i\right) + \left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right)\right) \\
&\quad \le \left(\tau_i + 2M_\Omega^2\right)\int_\Omega \left(h^i - \beta_\varepsilon^i(\vartheta_\mathcal{H}^i)\right)^2 + \frac{1}{4\tau_i}\int_\Omega \left(\vartheta_\varepsilon^i - \vartheta_\mathcal{H}^i\right)^2 + \frac{1}{8M_\Omega^2}\int_\Omega \left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right)^2 \\
&\quad \le c\int_\Omega \left(h^i - \beta_\varepsilon^i(\vartheta_\mathcal{H}^i)\right)^2 + \frac{1}{4\tau_i}\int_\Omega \left(\vartheta_\varepsilon^i - \vartheta_\mathcal{H}^i\right)^2 + \frac{1}{8}\int_\Omega \left|\nabla\left(\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i - \operatorname{Ln}_\varepsilon \vartheta_\mathcal{H}^i\right)\right|^2
\end{aligned}$$

Combining all these inequalities and taking Definition 3.14 of $\vartheta_\mathcal{H}^i$ as well as the first two inequalities of Proposition 3.16 into account, we obtain

$$\left\|\vartheta_\varepsilon^i\right\|_{L^2(\Omega)} + \left\|\operatorname{Ln}_\varepsilon \vartheta_\varepsilon^i\right\|_{H^1(\Omega)} + \left\|\left(\operatorname{Ln}_\varepsilon' \vartheta_\varepsilon^i\right)^{1/2}\nabla\vartheta_\varepsilon^i\right\|_{L^2(\Omega)} \le c \qquad (4.10)$$

A consequence We set $\Omega_{i,\varepsilon}^- := \{x \in \Omega : \vartheta_\varepsilon^i(x) \le 1\}$ and $\Omega_{i,\varepsilon}^+ = \{x \in \Omega : \vartheta_\varepsilon^i(x) > 1\}$. Then with inequalities (3.3c) yields

$$\begin{aligned}\int_\Omega \mathrm{Ln}_\varepsilon'(\vartheta_\varepsilon^i)\left|\nabla\vartheta_\varepsilon^i\right|^2 &\ge \int_{\Omega_{i,\varepsilon}^-} \mathrm{Ln}_\varepsilon'(\vartheta_\varepsilon^i)\left|\nabla\vartheta_\varepsilon^i\right|^2 \ge \int_{\Omega_{i,\varepsilon}^-}\left|\nabla\vartheta_\varepsilon^i\right|^2 \\ \int_\Omega \mathrm{Ln}_\varepsilon'(\vartheta_\varepsilon^i)\left|\nabla\vartheta_\varepsilon^i\right|^2 &\ge \int_{\Omega_{i,\varepsilon}^+} \mathrm{Ln}_\varepsilon'(\vartheta_\varepsilon^i)\left|\nabla\vartheta_\varepsilon^i\right|^2 \ge \int_{\Omega_{i,\varepsilon}^+}\frac{\left|\nabla\vartheta_\varepsilon^i\right|^2}{\vartheta_\varepsilon^i}.\end{aligned} \tag{4.11a}$$

Remarking that $\frac{\left|\nabla\vartheta_\varepsilon^i\right|^2}{\vartheta_\varepsilon^i} = \left|\nabla(\vartheta^i)_\varepsilon^{1/2}\right|^2$ and using the norm estimates (4.10) implies

$$\int_{\Omega_{i,\varepsilon}^-}\left|\nabla\vartheta_\varepsilon^i\right|^2 \le c \quad \text{and} \quad \int_{\Omega_{i,\varepsilon}^+}\left|\nabla(\vartheta_\varepsilon^i)^{1/2}\right|^2 \le c. \tag{4.11b}$$

Now, we can estimate $\nabla\vartheta_\varepsilon^i$ in a suitable norm. Accounting for the first estimate in (4.11b), we observe

$$\|\nabla\vartheta\|_{L^{4/3}(\Omega_{i,\varepsilon}^-)} \le c\,\|\nabla\vartheta\|_{L^2(\Omega_{i,\varepsilon}^-)} \le c. \tag{4.11c}$$

On the other hand using $\nabla\vartheta_\varepsilon^i = 2(\vartheta_\varepsilon^i)^{1/2}\nabla(\vartheta_\varepsilon^i)^{1/2}$ and Hölder's inequality with the exponents $1/4 + 1/2 = 3/4$ we obtain

$$\|\nabla\vartheta\|_{L^{4/3}(\Omega_{i,\varepsilon}^+)} \le 2\left\|(\vartheta_\varepsilon^i)^{1/2}\right\|_{L^4(\Omega_{i,\varepsilon}^+)}\left\|\nabla(\vartheta_\varepsilon^i)^{1/2}\right\|_{L^2(\Omega_{i,\varepsilon}^+)} \le 2\left\|\vartheta_\varepsilon^i\right\|_{L^2(\Omega)}^{1/2}\left\|\nabla(\vartheta_\varepsilon^i)^{1/2}\right\|_{L^2(\Omega_{i,\varepsilon}^+)} \le c \tag{4.11d}$$

by the estimates (4.10) and (4.11b).

Finally, we observe

$$\begin{aligned}&\left\|\nabla\vartheta_\varepsilon^i\right\|_{L^{4/3}(\Omega)}^{4/3} = \left\|\nabla\vartheta_\varepsilon^i\right\|_{L^{4/3}(\Omega_{i,\varepsilon}^-)}^{4/3} + \left\|\nabla\vartheta_\varepsilon^i\right\|_{L^{4/3}(\Omega_{i,\varepsilon}^+)}^{4/3} \quad \text{whence} \\ &\left\|\nabla\vartheta_\varepsilon^i\right\|_{L^{4/3}(\Omega)}^{4/3} \le c\left(\left\|\nabla\vartheta_\varepsilon^i\right\|_{L^{4/3}(\Omega_{i,\varepsilon}^-)}^2 + \left\|\nabla\vartheta_\varepsilon^i\right\|_{L^{4/3}(\Omega_{i,\varepsilon}^+)}^2\right)\end{aligned}$$

so that using (4.11c) and (4.11d), we conclude with

$$\left\|\nabla\vartheta_\varepsilon^i\right\|_{L^{4/3}(\Omega)} \le c. \tag{4.12}$$

Second a priori estimate We test (4.5a) with $\beta_\varepsilon^i(\vartheta_\varepsilon^i) - \beta_\varepsilon^i(\vartheta_{\mathcal{H}}^i)$. We point out the term involving the Laplacian

$$\int_\Omega \Delta(\mathrm{Ln}_\varepsilon\,\vartheta_\varepsilon^i)(\beta_\varepsilon^i(\vartheta_\varepsilon^i) - \beta_\varepsilon^i(\vartheta_{\mathcal{H}}^i)) = \int_\Omega \nabla(\mathrm{Ln}_\varepsilon\,\vartheta_\varepsilon^i)\cdot(\beta_\varepsilon^{i\prime}(\vartheta_\varepsilon^i)\nabla\vartheta_\varepsilon^i + \beta_{\varepsilon,x}^i(\vartheta_\varepsilon^i) - \nabla\beta_\varepsilon^i(\vartheta_{\mathcal{H}}^i)).$$

Integrating and using the above stated formula results in

$$\begin{aligned}&\frac{1}{\tau_i}\int_\Omega\left(\vartheta_\varepsilon^i - \vartheta_{\mathcal{H}}^i\right)\left(\beta_\varepsilon^i(\vartheta_\varepsilon^i) - \beta_\varepsilon^i(\vartheta_{\mathcal{H}}^i)\right) + \int_\Omega \nabla\left(\mathrm{Ln}_\varepsilon\,\vartheta_\varepsilon^i\right)\cdot\beta_\varepsilon^{i\prime}(\vartheta_\varepsilon^i)\nabla\vartheta_\varepsilon^i + \int_\Omega\left(\beta_\varepsilon^i(\vartheta_\varepsilon^i) - \beta_\varepsilon^i(\vartheta_{\mathcal{H}}^i)\right)^2 \\ &\quad = -\frac{1}{\tau_i}\int_\Omega \vartheta_{\mathcal{H}}^i\left(\beta_\varepsilon^i(\vartheta_\varepsilon^i) - \beta_\varepsilon^i(\vartheta_{\mathcal{H}}^i)\right) - \int_\Omega \nabla\,\mathrm{Ln}_\varepsilon\,\vartheta_\varepsilon^i\cdot\beta_{\varepsilon,x}^i(\vartheta_\varepsilon^i) + \int_\Omega \nabla\left(\mathrm{Ln}_\varepsilon\,\vartheta_\varepsilon^i\right)\cdot\nabla\beta_\varepsilon^i(\vartheta_{\mathcal{H}}^i) \\ &\qquad + \int_\Omega\left(h^i - \beta_\varepsilon^i(\vartheta_{\mathcal{H}}^i)\right)\left(\beta_\varepsilon^i(\vartheta_\varepsilon^i) - \beta_\varepsilon^i(\vartheta_{\mathcal{H}}^i)\right)\end{aligned}$$

All integrals on the left-hand side are again nonnegative. For the right-hand side we estimate term by term. For the first one it yields

$$\frac{1}{\tau_i}\int_\Omega \vartheta^i_{\mathcal{H}}\left(\beta^i_\varepsilon(\vartheta^i_\varepsilon)-\beta^i_\varepsilon(\vartheta^i_{\mathcal{H}})\right) \le c\left\|\vartheta^i_{\mathcal{H}}\right\|^2_{L^2(\Omega)} + \frac{1}{8}\int_\Omega \left(\beta^i_\varepsilon(\vartheta^i_\varepsilon)-\beta^i_\varepsilon(\vartheta^i_{\mathcal{H}})\right)^2,$$

where we stress out that c now depends also on τ_i. In the next integral we use Proposition 3.12

$$\begin{aligned}\int_\Omega \nabla \operatorname{Ln}_\varepsilon \vartheta^i_\varepsilon \cdot \beta^i_{\varepsilon,x}(\vartheta^i_\varepsilon) &\le \widetilde{c}_\delta \left\|\operatorname{Ln}_\varepsilon \vartheta^i_\varepsilon\right\|^2_{H^1(\Omega)} + \widetilde{\delta}\int_\Omega \left(1+\beta^i_\varepsilon(\vartheta^i_\varepsilon)\right)^2 \\ &\le c_\delta \left\|\operatorname{Ln}_\varepsilon \vartheta^i_\varepsilon\right\|^2_{H^1(\Omega)} + \delta\int_\Omega \left(\beta^i_\varepsilon(\vartheta^i_\varepsilon)-\beta^i_\varepsilon(\vartheta^i_{\mathcal{H}})\right)^2 + c.\end{aligned}$$

The following integral is estimated by using again Young's inequality

$$\int_\Omega \nabla\left(\operatorname{Ln}_\varepsilon \vartheta^i_\varepsilon\right)\cdot\nabla\beta^i_\varepsilon(\vartheta^i_{\mathcal{H}}) \le \frac{1}{2}\left\|\operatorname{Ln}_\varepsilon \vartheta^i_\varepsilon\right\|^2_{H^1(\Omega)} + \frac{1}{2}\left\|\beta^i_\varepsilon(\vartheta^i_{\mathcal{H}})\right\|^2_{H^1(\Omega)}.$$

Finally, we have achieved for the last integral

$$\int_\Omega \left(h^i-\beta^i_\varepsilon(\vartheta^i_{\mathcal{H}})\right)\left(\beta^i_\varepsilon(\vartheta^i_\varepsilon)-\beta^i_\varepsilon(\vartheta^i_{\mathcal{H}})\right) \le c\left\|h^i-\beta^i_\varepsilon(\vartheta^i_{\mathcal{H}})\right\|^2_{L^2(\Omega)} + \frac{1}{8}\int_\Omega\left(\beta^i_\varepsilon(\vartheta^i_\varepsilon)-\beta^i_\varepsilon(\vartheta^i_{\mathcal{H}})\right)^2.$$

Combining all these inequalities and choosing $\delta = 1/4$ leads to

$$\left\|\beta^i_\varepsilon(\vartheta^i_\varepsilon)-\beta^i_\varepsilon(\vartheta^i_{\mathcal{H}})\right\|^2_{L^2(\Omega)} \le c\left\|\vartheta^i_{\mathcal{H}}\right\|_{L^2(\Omega)} + c\left\|\operatorname{Ln}_\varepsilon \vartheta^i_\varepsilon\right\|^2_{H^1(\Omega)} + c\left\|\beta^i_\varepsilon(\vartheta^i_{\mathcal{H}})\right\|^2_{H^1(\Omega)} + c\left\|h^i\right\|^2_{L^2(\Omega)} + c.$$

Taking Proposition 3.16 as well as the norm estimate (4.10) into account we conclude with

$$\left\|\beta^i_\varepsilon(\vartheta^i_\varepsilon)\right\|_{L^2(\Omega)} \le c. \tag{4.14}$$

4.2.5 Passing to the limit in ε

The norm estimates (4.10), (4.12) and (4.14) lead to the following convergences for $\varepsilon \to 0$

$$\vartheta^i_\varepsilon \rightharpoonup \vartheta^i \quad \text{weakly in } L^2(\Omega) \tag{4.15a}$$
$$\operatorname{Ln}_\varepsilon \vartheta^i_\varepsilon \rightharpoonup \ell^i \quad \text{weakly in } H^1(\Omega) \tag{4.15b}$$
$$\nabla\vartheta^i_\varepsilon \rightharpoonup \nabla\vartheta^i \quad \text{weakly in } L^{4/3}(\Omega) \tag{4.15c}$$
$$\beta^i_\varepsilon(\vartheta^i_\varepsilon) \rightharpoonup \xi^i \quad \text{weakly in } L^2(\Omega) \tag{4.15d}$$

Considering (4.15a) and (4.15c) leads to $\vartheta^i_\varepsilon \rightharpoonup \vartheta^i$ weakly in $W^{1,4/3}(\Omega)$. Further, $W^{1,4/3}(\Omega)$ is compactly embedded in $L^2(\Omega)$ and consequently

$$\vartheta^i_\varepsilon \to \vartheta^i \quad \text{strongly in } L^2(\Omega). \tag{4.16}$$

Therewith, we obtain by (4.15b) $\operatorname{Ln}_\varepsilon \vartheta^i_\varepsilon \to \ell^i$ strongly in $L^2(\Omega)$. Now, by $\operatorname{Ln}_\varepsilon \vartheta^i_\varepsilon = \varepsilon\vartheta^i_\varepsilon + \ln_\varepsilon \vartheta^i_\varepsilon$ and $\varepsilon\vartheta^i_\varepsilon \to 0$ strongly in $L^2(\Omega)$ follows $\ln_\varepsilon \vartheta^i_\varepsilon \to \ell^i$ strongly in $L^2(\Omega)$. But, then it holds also the convergence of $\limsup_{k\to\infty} \vartheta^i_{\varepsilon_k} \ln_{\varepsilon_k} \vartheta^i_{\varepsilon_k} = \vartheta^i \ell^i$ a.e. in Ω. Since $\ln_{\varepsilon_k}$ is a Yosida approximation we can apply [4, Proposition 2.5, p. 27] and conclude

$$\ell^i \in \ln \vartheta^i \quad \text{and therefore } \vartheta^i > 0 \text{ and } \ell^i = \ln \vartheta^i. \tag{4.17}$$

Finally, from Lemma 3.13 it follows that $\xi^i = \beta^i(\vartheta^i)$ and the remaining terms involving F' and G' can be identified with their limits due to the convergence of ϑ^i_ε (4.16) and the assumptions (2.2b).

5 Convergence to continuous solution

Take a discrete solution $(\vartheta^{\mathcal{P}}, \chi^{\mathcal{P}}, \xi^{\mathcal{P}})$ with $\xi^i = \beta^i(\vartheta^i)$ for all $i = 1, \dots, N$ satisfying

$$\frac{\vartheta^i - \vartheta^{i-1}}{\tau_i} - G'(\chi^i)\frac{\chi^i - \chi^{i-1}}{\tau_i} - \Delta \ln \vartheta^i + \beta^i(\vartheta^i) = \overline{\pi}^i(\vartheta^{i-1}) \tag{5.1a}$$

$$\frac{\chi^i - \chi^{i-1}}{\tau_i} - \Delta\chi^i + F'(\chi^i) + G'(\chi^i)\vartheta^{i-1} = 0 \qquad \text{for a.a. } x \in \Omega \tag{5.1b}$$

5.1 Estimates uniform with respect to τ

5.1.1 First a priori estimate

We add the first equation (5.1a) tested by $\vartheta^i - \vartheta^i_{\mathcal{H}} + \delta\left(\ln \vartheta^i - \ln \vartheta^i_{\mathcal{H}}\right)$ and the second equation (5.1b) tested by $\dfrac{1}{\tau_i}\left(\chi^i - \chi^{i-1}\right)$

$$\begin{aligned}
&\int_\Omega \frac{1}{\tau_i}\left[\frac{1}{2}\left(\vartheta^i - \vartheta^{i-1}\right)^2 + \frac{1}{2}\left(\vartheta^i - \vartheta^i_{\mathcal{H}}\right)^2 - \frac{1}{2}\left(\vartheta^{i-1} - \vartheta^i_{\mathcal{H}}\right)^2\right] \\
&\quad + \frac{\delta}{\tau_i}\int_\Omega \left(\vartheta^i - \vartheta^{i-1}\right)\ln\vartheta^i + \int_\Omega \nabla\ln\vartheta^i\cdot\nabla\vartheta^i + \delta\int_\Omega \left|\nabla\left(\ln\vartheta^i - \ln\vartheta^i_{\mathcal{H}}\right)\right|^2 \\
&\quad + \int_\Omega \left(\beta^i(\vartheta^i) - \beta^i(\vartheta^i_{\mathcal{H}})\right)\left(\left(\vartheta^i - \vartheta^i_{\mathcal{H}}\right) + \delta\left(\ln\vartheta^i - \ln\vartheta^i_{\mathcal{H}}\right)\right) \\
&\quad + \int_\Omega \frac{\left(\chi^i - \chi^{i-1}\right)^2}{(\tau_i)^2} + \frac{1}{\tau_i}\int_\Omega\left[\frac{1}{2}\left|\nabla\chi^i\right|^2 + \frac{1}{2}\left|\nabla\left(\chi^i - \chi^{i-1}\right)\right|^2 - \frac{1}{2}\left|\nabla\chi^{i-1}\right|^2\right] \\
&= \frac{\delta}{\tau_i}\int_\Omega\left(\vartheta^i - \vartheta^{i-1}\right)\ln\vartheta^i_{\mathcal{H}} + \int_\Omega \nabla\ln\vartheta^i\cdot\nabla\vartheta^i_{\mathcal{H}} - \delta\int_\Omega \nabla\ln\vartheta^i_{\mathcal{H}}\cdot\nabla\left(\ln\vartheta^i - \ln\vartheta^i_{\mathcal{H}}\right) \\
&\quad + \int_\Omega G'(\chi^i)\frac{\chi^i - \chi^{i-1}}{\tau_i}\left(\vartheta^i - \vartheta^{i-1} - \vartheta^i_{\mathcal{H}}\right) + \delta\int_\Omega G'(\chi^i)\frac{\chi^i - \chi^{i-1}}{\tau_i}\left(\ln(\vartheta^i) - \ln(\vartheta^i_{\mathcal{H}})\right) \\
&\quad + \int_\Omega\left(\overline{\pi}^i(\vartheta^{i-1}) - \beta^i(\vartheta^i_{\mathcal{H}})\right)\left(\vartheta^i - \vartheta^i_{\mathcal{H}}\right) + \delta\int_\Omega\left(\overline{\pi}^i(\vartheta^{i-1}) - \beta^i(\vartheta^i_{\mathcal{H}})\right)\left(\ln\vartheta^i - \ln\vartheta^i_{\mathcal{H}}\right) \\
&\quad - \int_\Omega F'(\chi^i)\frac{\chi^i - \chi^{i-1}}{\tau_i}.
\end{aligned} \tag{5.2}$$

We add the term with the minus sign in the first line to the inequality and estimate it

$$\frac{1}{2\tau_i}\int_\Omega \left(\vartheta^{i-1}-\vartheta^i_{\mathcal{H}}\right)^2 \leq \frac{1}{\tau_i}\left\|\vartheta^{i-1}-\vartheta^{i-1}_{\mathcal{H}}\right\|^2_{L^2(\Omega)} + \frac{1}{8\tau_i}\left\|\vartheta^i_{\mathcal{H}}-\vartheta^{i-1}_{\mathcal{H}}\right\|^2_{L^2(\Omega)}.$$

For a lower bound of the integral in the second line we us the monotonicity of the primitive of ln given by $\int_1^r \ln s \,\mathrm{d}s = r(\ln r - 1) + 1$

$$\frac{\delta}{\tau_i}\int_\Omega \left(\vartheta^i-\vartheta^{i-1}\right)\ln\vartheta^i \geq \frac{\delta}{\tau_i}\int_\Omega\int_{\vartheta^{i-1}}^{\vartheta^i}\ln s\,\mathrm{d}s = \int_\Omega \left(\vartheta^i\left(\ln\vartheta^i-1\right)-\vartheta^{i-1}\left(\ln\vartheta^{i-1}-1\right)\right).$$

Now, we integrate the first integral on the right-hand side discretely by parts

$$\begin{aligned}\frac{\delta}{\tau_i}\int_\Omega \left(\vartheta^i-\vartheta^{i-1}\right)\ln\vartheta^i_{\mathcal{H}} = \frac{\delta}{\tau_i}\int_\Omega \big[&(\vartheta^i-\vartheta^i_{\mathcal{H}})\ln\vartheta^i_{\mathcal{H}} - (\vartheta^{i-1}-\vartheta^{i-1}_{\mathcal{H}})(\ln\vartheta^i_{\mathcal{H}}-\ln\vartheta^{i-1}_{\mathcal{H}}) \\ &- (\vartheta^{i-1}-\vartheta^{i-1}_{\mathcal{H}})\ln\vartheta^{i-1}_{\mathcal{H}} + (\vartheta^i_{\mathcal{H}}-\vartheta^{i-1}_{\mathcal{H}})\ln\vartheta^i_{\mathcal{H}}\big].\end{aligned}$$

By using Young's inequality the next integral becomes

$$\begin{aligned}\int_\Omega \nabla\ln\vartheta^i\cdot\nabla\vartheta^i_{\mathcal{H}} &= \int_\Omega \nabla\left(\ln\vartheta^i-\ln\vartheta^i_{\mathcal{H}}\right)\cdot\nabla\vartheta^i_{\mathcal{H}} + \int_\Omega \nabla\ln\vartheta^i_{\mathcal{H}}\cdot\nabla\vartheta^i_{\mathcal{H}} \\ &\leq \frac{\delta}{8}\int_\Omega \left|\nabla\left(\ln\vartheta^i-\ln\vartheta^i_{\mathcal{H}}\right)\right|^2 + \frac{4}{\delta}\left\|\nabla\vartheta^i_{\mathcal{H}}\right\|^2_{L^2(\Omega)} + \frac{\delta}{8}\left\|\nabla\ln\vartheta^i_{\mathcal{H}}\right\|^2_{L^2(\Omega)}.\end{aligned}$$

Similarly, it follows that

$$-\delta\int_\Omega \nabla\ln\vartheta^i_{\mathcal{H}}\nabla\left(\ln\vartheta^i-\ln\vartheta^i_{\mathcal{H}}\right) \leq \frac{\delta}{8}\int_\Omega \left|\nabla\left(\ln\vartheta^i-\ln\vartheta^i_{\mathcal{H}}\right)\right|^2 + \frac{2}{\delta}\left\|\nabla\ln\vartheta^i_{\mathcal{H}}\right\|^2_{L^2(\Omega)}.$$

For the next both integrals we use the boundedness of G' by L_G (2.8) and get

$$\begin{aligned}&\int_\Omega G'(\chi^i)\frac{\chi^i-\chi^{i-1}}{\tau_i}\left(\vartheta^i-\vartheta^{i-1}-\vartheta^i_{\mathcal{H}}\right) \\ &\quad\leq \frac{1}{4}\int_\Omega \left(\frac{\chi^i-\chi^{i-1}}{\tau_i}\right)^2 + L_G^2\int_\Omega \left(\vartheta^i-\vartheta^{i-1}\right)^2 + L_G^2\left\|\vartheta^i_{\mathcal{H}}\right\|^2_{L^2(\Omega)}.\end{aligned}$$

The second integral can be estimated with the help of the Poincaré inequality as follows

$$\begin{aligned}&\delta\int_\Omega G'(\chi^i)\frac{\chi^i-\chi^{i-1}}{\tau_i}\left(\ln(\vartheta^i)-\ln(\vartheta^i_{\mathcal{H}})\right) \\ &\quad\leq \frac{\delta}{8}\int_\Omega \left|\nabla\left(\ln(\vartheta^i)-\ln(\vartheta^i_{\mathcal{H}})\right)\right|^2 + 2\delta M_\Omega^2 L_G^2\int_\Omega \left(\frac{\chi^i-\chi^{i-1}}{\tau_i}\right)^2.\end{aligned}$$

The assumptions on π pointed out in (2.2j) allow the estimate

$$\begin{aligned}\int_\Omega \left(\overline{\pi}^i(\vartheta^{i-1})-\beta^i(\vartheta^i_{\mathcal{H}})\right)\left(\vartheta^i-\vartheta^i_{\mathcal{H}}\right) &\leq \int_\Omega \left(\left|\beta^i(\vartheta^i_{\mathcal{H}})\right| + L_\pi\left|\vartheta^{i-1}\right| + \overline{\pi}^i_0\right)\left|\vartheta^i-\vartheta^i_{\mathcal{H}}\right| \\ &\leq \int_\Omega \left(\vartheta^i-\vartheta^i_{\mathcal{H}}\right)^2 + 2L_\pi^2\int_\Omega \left(\vartheta^{i-1}-\vartheta^{i-1}_{\mathcal{H}}\right)^2 + 2R,\end{aligned}$$

where $R := \|\beta^i(\vartheta^i_{\mathcal{H}})\|^2_{L^2(\Omega)} + L_\pi^2 \left\|\vartheta^{i-1}_{\mathcal{H}}\right\|^2_{L^2(\Omega)} + \left\|\overline{\pi}^i_0\right\|^2_{L^2(\Omega)}$ is a remainder term. In a similar way it yields

$$\delta \int_\Omega \left(\overline{\pi}^i(\vartheta^{i-1}) - \beta^i(\vartheta^i_{\mathcal{H}})\right)\left(\ln\vartheta^i - \ln\vartheta^i_{\mathcal{H}}\right) \\ \leq \frac{\delta}{8}\int_\Omega \left|\nabla\left(\ln\vartheta^i - \ln\vartheta^i_{\mathcal{H}}\right)\right|^2 + 4\delta M_\Omega^2 L_\pi^2 \int_\Omega \left(\vartheta^{i-1} - \vartheta^{i-1}_{\mathcal{H}}\right)^2 + 4\delta M_\Omega^2 R$$

whereby R is defined like above. Finally, the last integral becomes with the boundedness of F'

$$\int_\Omega F'(\chi^i)\frac{\chi^i - \chi^{i-1}}{\tau_i} \leq \frac{1}{2}\int_\Omega \left(\frac{\chi^i - \chi^{i-1}}{\tau_i}\right)^2 + \frac{L_F^2}{2}.$$

We put all the norms pointed out in Proposition 3.16 with respect to $\vartheta^i_{\mathcal{H}}$ and also the constants with respect to G, F as well as the bounded norm $\left\|\overline{\pi}^i_0\right\|_{L^2(\Omega)}$ in one constant $C > 0$. Now, we combine all these inequalities and multiply the resulting inequality by τ_i. Further, we sum up from 1 to m with $1 \leq m \leq N$ and choose τ and δ small enough

$$\frac{1}{4}\sum_{i=1}^m \left\|\vartheta^i - \vartheta^{i-1}\right\|^2_{L^2(\Omega)} + \frac{1}{4}\left\|\vartheta^m - \vartheta^m_{\mathcal{H}}\right\|^2_{L^2(\Omega)} + \int_\Omega \left(\vartheta^m(\ln\vartheta^m - 1) + 1\right) \\ + \left\|\vartheta_\tau^{-1/2}\nabla\vartheta_\tau\right\|^2_{L^2(0,t_m;L^2(\Omega))} + \frac{1}{8}\left\|\nabla\left(\ln\vartheta_\tau - \ln\vartheta_{\mathcal{H},\tau}\right)\right\|^2_{L^2(0,t_m;L^2(\Omega))} \\ + \frac{1}{4}\left\|\partial_t\widehat{\chi}_\tau\right\|^2_{L^2(0,t_m;L^2(\Omega))} + \frac{1}{2}\left\|\nabla\chi^m\right\|^2_{L^2(\Omega)} + \frac{1}{2}\sum_{i=1}^m \left\|\nabla\left(\chi^i - \chi^{i-1}\right)\right\|^2_{L^2(\Omega)} \\ \leq 2\sum_{i=0}^{m-1}\left\|\vartheta^i - \vartheta^i_{\mathcal{H}}\right\|^2_{L^2(\Omega)} + \int_\Omega \left(\vartheta^0(\ln\vartheta^0 - 1) + 1\right) + \frac{1}{2}\left\|\nabla\chi^0\right\|^2_{L^2(\Omega)} + Ct_m \\ + \frac{1}{8}\sum_{i=1}^m \left\|\vartheta^i_{\mathcal{H}} - \vartheta^{i-1}_{\mathcal{H}}\right\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega (\vartheta^m - \vartheta^m_{\mathcal{H}})\ln\vartheta^m_{\mathcal{H}} - \frac{1}{2}\int_\Omega \left(\vartheta^0 - \vartheta^0_{\mathcal{H}}\right)\ln\vartheta^0_{\mathcal{H}} \\ - \frac{1}{2}\sum_{i=0}^{m-1}\int_\Omega \left(\vartheta^i - \vartheta^i_{\mathcal{H}}\right)\left(\ln\vartheta^{i+1}_{\mathcal{H}} - \ln\vartheta^i_{\mathcal{H}}\right) + \frac{1}{2}\sum_{i=1}^m \int_\Omega \left(\vartheta^i_{\mathcal{H}} - \vartheta^{i-1}_{\mathcal{H}}\right)\ln\vartheta^i_{\mathcal{H}}$$

We observe the following estimates

$$\int_\Omega (\vartheta^m - \vartheta^m_{\mathcal{H}})\ln\vartheta^m_{\mathcal{H}} \leq \frac{1}{4}\left\|\vartheta^m - \vartheta^m_{\mathcal{H}}\right\|^2_{L^2(\Omega)} + \left\|\ln\vartheta^m_{\mathcal{H}}\right\|^2_{L^2(\Omega)} \\ \int_\Omega \left(\vartheta^i - \vartheta^i_{\mathcal{H}}\right)\left(\ln\vartheta^{i+1}_{\mathcal{H}} - \ln\vartheta^i_{\mathcal{H}}\right) \leq \frac{1}{2}\left\|\vartheta^i - \vartheta^i_{\mathcal{H}}\right\|^2_{L^2(\Omega)} + \frac{1}{2\vartheta_*^2}\int_\Omega \left(\vartheta^i_{\mathcal{H}} - \vartheta^{i-1}_{\mathcal{H}}\right)^2$$

where we used the boundedness of $\vartheta_{\mathcal{H},\tau}$ and thus the Lipschitz continuity of ln on $[\vartheta_*, \vartheta^*]$. The very last integral is estimated with Young's inequality

$$\int_\Omega \left(\vartheta^i_{\mathcal{H}} - \vartheta^{i-1}_{\mathcal{H}}\right)\ln\vartheta^i_{\mathcal{H}} \leq \frac{1}{2\tau_i}\int_\Omega \left(\vartheta^i_{\mathcal{H}} - \vartheta^{i-1}_{\mathcal{H}}\right)^2 + \frac{\tau_i}{2}\int_\Omega \left(\ln\vartheta^i_{\mathcal{H}}\right)^2.$$

We collect all further norms also depending on the initial values and put them into the constant C. We have supposed a uniform partition $\mathcal{P}$ and thus yields $\sigma\tau \leq \tau_i \leq \tau$ with $\sigma \in (0,1]$ for all $1 \leq i \leq N$. Hence, we obtain the estimate

$$\begin{aligned}
&\frac{\sigma\tau}{8}\sum_{i=1}^{m}\tau_i\left\|\frac{\vartheta^i-\vartheta^{i-1}}{\tau_i}\right\|_{L^2(\Omega)}^2+\frac{1}{4}\left\|\vartheta^m-\vartheta^m_{\mathcal{H}}\right\|_{L^2(\Omega)}^2+\int_\Omega\left(\vartheta^m(\ln\vartheta^m-1)+1\right)\\
&\quad+\left\|(\vartheta_\tau)^{-\frac{1}{2}}\nabla\vartheta_\tau\right\|_{L^2(0,t_m;L^2(\Omega))}^2+\frac{1}{8}\left\|\nabla\left(\ln\vartheta_\tau-\ln\vartheta_{\mathcal{H},\tau}\right)\right\|_{L^2(0,t_m;L^2(\Omega))}^2\\
&\quad+\frac{1}{4}\left\|\partial_t\widehat{\chi}_\tau\right\|_{L^2(0,t_m;L^2(\Omega))}^2+\frac{1}{2}\left\|\nabla\chi^m\right\|_{L^2(\Omega)}^2+\frac{\sigma\tau}{2}\sum_{i=1}^{m}\tau_i\left\|\nabla\frac{\chi^i-\chi^{i-1}}{\tau_i}\right\|_{L^2(\Omega)}^2\\
&\leq 3\sum_{i=0}^{m-1}\left\|\vartheta^i-\vartheta^i_{\mathcal{H}}\right\|_{L^2(\Omega)}^2+\sum_{i=1}^{m}\tau_i\left\|\frac{\vartheta^i_{\mathcal{H}}-\vartheta^{i-1}_{\mathcal{H}}}{\tau_i}\right\|_{L^2(\Omega)}^2+\sum_{i=1}^{m}\tau_i\left\|\ln\vartheta_{\mathcal{H}}\right\|_{L^2(\Omega)}^2+C(t_m+1),
\end{aligned}$$

The last step is applying the discrete version of Gronwall's Lemma for controlling the term $\|\vartheta^m-\vartheta^m_{\mathcal{H}}\|_{L^2(\Omega)}$. Finally, using the Poincaré inequality, we have concluded the following norm estimates

$$\begin{aligned}
&\tau\left\|\partial\widehat{\vartheta}_\tau\right\|_{L^2(0,T;L^2(\Omega))}^2+\left\|\vartheta_\tau\right\|_{L^\infty(0,T;L^2(\Omega))}^2+\sup_{t\in[0,T]}\int_\Omega\left(\vartheta_\tau(\ln\vartheta_\tau-1)+1\right)\\
&\quad+\left\|\vartheta_\tau^{-1/2}\nabla\vartheta_\tau\right\|_{L^2(0,T;L^2(\Omega))}^2+\left\|\ln\vartheta_\tau\right\|_{L^2(0,T;H^1(\Omega))}^2\\
&\quad+\left\|\partial_t\widehat{\chi}_\tau\right\|_{L^2(0,T;L^2(\Omega))}^2+\left\|\chi_\tau\right\|_{L^\infty(0,T;H^1(\Omega))}^2+\tau\left\|\partial_t\widehat{\chi}_\tau\right\|_{L^2(0,T;H^1(\Omega))}^2\\
&\leq\left\|\partial\widehat{\vartheta}_{\mathcal{H},\tau}\right\|_{L^2(0,T;L^2(\Omega)}^2+\left\|\ln\vartheta_{\mathcal{H},\tau}\right\|_{L^2(0,T;L^2(\Omega)}^2+C\leq C,
\end{aligned}\tag{5.3}$$

due to the boundedness of $\vartheta_{\mathcal{H},\tau}$ in suitable norms given in Proposition 3.16.

Consequences By comparing in (4.1a) we achieve

$$\widehat{\vartheta}_\tau\in H^1(0,T;H^{-1}(\Omega))\quad\text{with}\quad\left\|\partial\widehat{\vartheta}_\tau\right\|_{L^2(0,T;H^{-1}(\Omega))}\leq C.\tag{5.4}$$

Also in comparison to (4.1b) we can even conclude

$$\chi_\tau\in L^2(0,T;H^2(\Omega))\quad\text{with}\quad\left\|\chi_\tau\right\|_{L^2(0,T;H^2(\Omega))}\leq C.\tag{5.5}$$

Additionally, we can prove in the same way already shown for (4.12)

$$\nabla\vartheta_\tau\in L^2(0,T;L^{4/3}(\Omega))\quad\text{with}\quad\left\|\nabla\vartheta_\tau\right\|_{L^2(0,T;L^{4/3}(\Omega))}\leq C.\tag{5.6}$$

5.1.2 Second a priori estimate

We are testing the equation (4.1a) by $\beta^i(\vartheta^i)-\beta^i(\vartheta^i_{\mathcal{H}})$. The term $\beta^i(\vartheta^i_{\mathcal{H}})$ allows us to integrate by parts without a remainder. The integral with the Laplacian can be estimated in an analogous way as in the second a priori estimate for the limit in ε

$$\int_\Omega\Delta(\ln\vartheta^i)(\beta^i(\vartheta^i)-\beta^i(\vartheta^i_{\mathcal{H}}))=\int_\Omega\nabla(\ln\vartheta^i)\cdot\left[\beta^{i\prime}(\vartheta^i)\nabla\vartheta^i+\beta^i_{,x}(\vartheta^i)-\nabla\beta^i(\vartheta^i_{\mathcal{H}})\right].$$

We obtain by integrating and using the above stated expression

$$\begin{aligned}
&\frac{1}{\tau_i}\int_\Omega \left(\vartheta^i-\vartheta^{i-1}\right)\beta^i(\vartheta^i)+\int_\Omega \nabla\left(\ln\vartheta^i\right)\cdot\beta^{i\prime}(\vartheta^i)\nabla\vartheta^i+\int_\Omega\left(\beta^i(\vartheta^i)-\beta^i(\vartheta^i_{\mathcal{H}})\right)^2\\
&=\frac{1}{\tau_i}\int_\Omega \left(\vartheta^i-\vartheta^{i-1}\right)\beta^i(\vartheta^i_{\mathcal{H}})-\int_\Omega \nabla\ln\vartheta^i\cdot\beta^i_{,x}(\vartheta^i)+\int_\Omega\nabla\left(\ln\vartheta^i\right)\cdot\nabla\beta^i(\vartheta^i_{\mathcal{H}}) \qquad (5.7)\\
&\quad+\int_\Omega\left(G'(\chi^i)\frac{\chi^i-\chi^{i-1}}{\tau_i}-\beta^i(\vartheta^i_{\mathcal{H}})+\overline{\pi}^i(\vartheta^{i-1})\right)\left(\beta^i(\vartheta^i)-\beta^i(\vartheta^i_{\mathcal{H}})\right).
\end{aligned}$$

We define $B^i(r):=\int_1^r\beta^i(s)\,\mathrm{d}s$, the primitive of β. Then, observing by (2.2g) that $\beta(x,t,\cdot)$ is nondecreasing, it turns out that B^i is nonnegative, due to (2.2i) and convex with respect to r. For the first integral we argue with the monotonicity of β^i

$$\frac{1}{\tau_i}\int_\Omega \left(\vartheta^i-\vartheta^{i-1}\right)\beta^i(\vartheta^i)\geq\frac{1}{\tau_i}\int_\Omega\int_{\vartheta^{i-1}}^{\vartheta^i}\beta^i(s)\,\mathrm{d}s=\frac{1}{\tau_i}\int_\Omega\left(B^i(\vartheta^i)-B^{i-1}(\vartheta^{i-1})\right).$$

Now, we handle the right-hand side. We integrate discretely by parts the first integral

$$\int_\Omega \left(\vartheta^i-\vartheta^{i-1}\right)\beta^i(\vartheta^i_{\mathcal{H}})=\int_\Omega\left[\vartheta^i\beta^i(\vartheta^i_{\mathcal{H}})-\vartheta^{i-1}\left(\beta^i(\vartheta^i_{\mathcal{H}})-\beta^{i-1}(\vartheta^{i-1}_{\mathcal{H}})\right)-\vartheta^{i-1}\beta^{i-1}(\vartheta^{i-1}_{\mathcal{H}})\right].$$

Next, we observe

$$\begin{aligned}
-\int_\Omega\nabla\ln\vartheta^i\cdot\beta^i_{,x}(\vartheta^i)&\leq 4\left\|\nabla\ln\vartheta^i\right\|^2_{L^2(\Omega)}+\frac{1}{16}\int_\Omega\left(\beta^i(\vartheta^i)+1\right)^2\\
&\leq 4\left\|\ln\vartheta^i\right\|^2_{H^1(\Omega)}+\frac{1}{8}\int_\Omega\left(\beta^i(\vartheta^i)-\beta^i(\vartheta^i_{\mathcal{H}})\right)^2+\frac{1}{8}\left\|\beta^i(\vartheta^i_{\mathcal{H}})+1\right\|^2_{L^2(\Omega)}.
\end{aligned}$$

The following integral becomes by Young's inequality

$$\int_\Omega\nabla\left(\ln\vartheta^i\right)\cdot\nabla\beta^i(\vartheta^i_{\mathcal{H}})\leq\frac{1}{2}\left\|\ln\vartheta^i\right\|^2_{H^1(\Omega)}+\frac{1}{2}\left\|\beta^i(\vartheta^i_{\mathcal{H}})\right\|^2_{H^1(\Omega)}.$$

Finally, it yields for the last integral

$$\begin{aligned}
&\int_\Omega\left(G'(\chi^i)\frac{\chi^i-\chi^{i-1}}{\tau_i}-\beta^i(\vartheta^i_{\mathcal{H}})+\overline{\pi}^i(\vartheta^{i-1})\right)\left(\beta^i(\vartheta^i)-\beta^i(\vartheta^i_{\mathcal{H}})\right)\\
&\leq 4L_G^2\left\|\frac{\chi^i-\chi^{i-1}}{\tau_i}\right\|^2_{L^2(\Omega)}+4\left\|\beta^i(\vartheta^i_{\mathcal{H}})\right\|^2_{L^2(\Omega)}+4\left\|\overline{\pi}^i(\vartheta^{i-1})\right\|^2_{L^2(\Omega)}+\frac{1}{8}\int_\Omega\left(\beta^i(\vartheta^i)-\beta^i(\vartheta^i_{\mathcal{H}})\right)^2,
\end{aligned}$$

where we actually remark $\left\|\overline{\pi}^i(\vartheta^{i-1})\right\|^2_{L^2(\Omega)}\leq L_\pi^2\left\|\vartheta^{i-1}\right\|^2_{L^2(\Omega)}+\left\|\overline{\pi}^i_0\right\|^2_{L^2(\Omega)}$. All these inequalities combined together result in

$$\begin{aligned}
&\frac{1}{\tau_i}\int_\Omega\left(B^i(\vartheta^i)-B^{i-1}(\vartheta^{i-1})\right)+\frac{1}{2}\left\|\beta^i(\vartheta^i)-\beta^i(\vartheta^i_{\mathcal{H}})\right\|^2_{L^2(\Omega)}\\
&\leq\frac{1}{\tau_i}\int_\Omega\left[\vartheta^i\beta^i(\vartheta^i_{\mathcal{H}})-\vartheta^{i-1}\left(\beta^i(\vartheta^i_{\mathcal{H}})-\beta^{i-1}(\vartheta^{i-1}_{\mathcal{H}})\right)-\vartheta^{i-1}\beta^{i-1}(\vartheta^{i-1}_{\mathcal{H}})\right]\\
&\quad+5\left\|\ln\vartheta^i\right\|^2_{H^1(\Omega)}+\left\|\beta^i(\vartheta^i_{\mathcal{H}})\right\|^2_{L^2(\Omega)}+L_\pi^2\left\|\vartheta^{i-1}\right\|^2_{L^2(\Omega)}+4L_G^2\left\|\frac{\chi^i-\chi^{i-1}}{\tau_i}\right\|^2_{L^2(\Omega)}+C,
\end{aligned}$$

where C depends on Ω and π_0. Now, multiplying by τ_i and summing up from 1 to N let the following estimate hold

$$\begin{aligned}
&\int_\Omega B^N(\vartheta^N) + \frac{1}{2}\left\|\beta_\tau(\vartheta_\tau) - \beta_\tau(\vartheta_{\mathcal{H},\tau})\right\|^2_{L^2(0,T;L^2(\Omega))} \\
&\leq \int_\Omega \vartheta^N \beta^N(\vartheta^N_{\mathcal{H}}) - \sum_{i=1}^{N-1} \tau_i \int_\Omega \vartheta^i \frac{\beta^{i+1}(\vartheta^{i+1}_{\mathcal{H}}) - \beta^i(\vartheta^i_{\mathcal{H}})}{\tau_i} + \int_\Omega \vartheta^0 \beta^0(\vartheta^0_{\mathcal{H}}) \\
&\quad + \int_\Omega B^0(\vartheta^0) + 5\left\|\ln \vartheta_\tau\right\|^2_{L^2(0,T;H^1(\Omega))} + \left\|\beta_\tau(\vartheta_{\mathcal{H},\tau})\right\|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad + L_\pi^2 \left\|\vartheta_\tau\right\|^2_{L^2(0,T;L^2(\Omega)} + 4L_G^2 \left\|\partial\widehat{\chi}_\tau\right\|^2_{L^2(0,T;L^2(\Omega))} + C,
\end{aligned}$$

where C also depends on the initial values and the endtime T. We observe the following estimates

$$\begin{aligned}
\int_\Omega \vartheta^N \beta^N(\vartheta^N_{\mathcal{H}}) &\leq \left\|\vartheta_\tau\right\|^2_{L^\infty(0,T;L^2(\Omega))} + \left\|\beta_\tau(\vartheta_{\mathcal{H},\tau})\right\|^2_{L^\infty(0,T;L^2(\Omega))} \\
\sum_{i=1}^{N-1} \tau_i \int_\Omega \vartheta^i \frac{\beta^{i+1}(\vartheta^{i+1}_{\mathcal{H}}) - \beta^i(\vartheta^i_{\mathcal{H}})}{\tau_i} &\leq \left\|\vartheta_\tau\right\|^2_{L^2(0,T;L^2(\Omega))} + \left\|\partial_t\widehat{\beta}_\tau(\vartheta_{\mathcal{H},\tau})\right\|^2_{L^2(0,T;L^2(\Omega)} \\
\int_\Omega \vartheta^0 \beta^0(\vartheta^0_{\mathcal{H}}) &\leq \left\|\vartheta_0\right\|^2_{L^2(\Omega)} + \left\|\beta_\tau(\vartheta_{\mathcal{H},\tau})\right\|_{L^\infty(0,T;L^2(\Omega))} \\
\int_\Omega B^0(\vartheta^0) &\leq \int_\Omega \beta_0(\vartheta^0) \leq \beta_0(\vartheta^*)\left|\Omega\right|.
\end{aligned}$$

Finally, we can conclude thanks to (5.3) and again Proposition 3.16

$$\left\|\beta_\tau(\vartheta_\tau)\right\|_{L^2(0,T;L^2(\Omega))} \leq C. \tag{5.8}$$

5.2 Passing to the limit in τ

The norm estimates (5.3) and (5.8) imply the existence of the following limits (possibly taking subsequences)

$$\begin{aligned}
\vartheta_\tau &\overset{*}{\rightharpoonup} \vartheta && \text{weakly star in } L^\infty(0,T;L^2(\Omega)) && \text{(5.9a)} \\
\nabla\vartheta_\tau &\rightharpoonup \nabla\vartheta && \text{weakly in } L^2(0,T;L^{4/3}(\Omega)) && \text{(5.9b)} \\
\widehat{\vartheta}_\tau &\overset{*}{\rightharpoonup} \widehat{\vartheta} && \text{weakly star in } L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)) && \text{(5.9c)} \\
\ln\vartheta_\tau &\rightharpoonup \ell && \text{weakly in } L^2(0,T;H^1(\Omega)) && \text{(5.9d)} \\
\chi_\tau &\overset{*}{\rightharpoonup} \chi && \text{weakly star in } L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)) && \text{(5.9e)} \\
\widehat{\chi}_\tau &\overset{*}{\rightharpoonup} \widehat{\chi} && \text{weakly star in } L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)) && \text{(5.9f)} \\
\beta_\tau(\vartheta_\tau) &\rightharpoonup \xi && \text{weakly in } L^2(0,T;L^2(\Omega)) && \text{(5.9g)}
\end{aligned}$$

as $\tau \to 0$. The notation $\tau \to 0$ means that there exists a family of partitions $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ with $\tau_n \to 0$ as $n \to \infty$.

Firstly, we estimate the difference $\widehat{\vartheta}_\tau - \vartheta_\tau$

$$\left\|\widehat{\vartheta}_\tau - \vartheta_\tau\right\|^2_{L^2(0,T;L^2(\Omega))} \leq \sum_{i=1}^{N} \int_{t_{i-1}}^{t^i} (t-t_i)^2 \left\|\frac{\vartheta^i - \vartheta^{i-1}}{\tau_i}\right\|^2_{L^2(\Omega)} \leq \tau^2 \left\|\partial\widehat{\vartheta}_\tau\right\|^2_{L^2(0,T;L^2(\Omega))}.$$

We do the same for the difference $\widehat{\chi}_\tau - \chi_\tau$

$$\left\|\widehat{\chi}_\tau - \chi_\tau\right\|^2_{L^2(0,T;H^1(\Omega))} \leq \sum_{i=1}^{N} \int_{t_{i-1}}^{t^i} (t-t_i)^2 \left\|\frac{\chi^i - \chi^{i-1}}{\tau_i}\right\|^2_{H^1(\Omega)} \leq \tau^2 \left\|\partial\widehat{\chi}_\tau\right\|^2_{L^2(0,T;H^1(\Omega))}.$$

Thanks to (5.3) we have

$$\left\|\widehat{\vartheta}_\tau - \vartheta_\tau\right\|_{L^2(0,T;L^2(\Omega))} \leq C\sqrt{\tau} \quad \text{and} \quad \left\|\widehat{\chi}_\tau - \chi_\tau\right\|_{L^2(0,T;H^1(\Omega))} \leq C\sqrt{\tau} \tag{5.10a}$$

As a consequence we can derive from (5.9c) and (5.9f)

$$\vartheta = \widehat{\vartheta} \quad \text{in } L^\infty(0,T;L^2(\Omega)) \quad \text{and} \quad \chi = \widehat{\chi} \quad \text{in } L^\infty(0,T;H^1(\Omega)). \tag{5.11}$$

Using the same arguments as in the limit in ε, particularly by the fact $(4/3)^* = 12/5 > 2$ and also by the Gagliardo-Nirenberg-Sobolev inequality, we observe that $W^{1,4/3}(\Omega)$ is compactly embedded in $L^2(\Omega)$ and thus (cf., e.g., [17, p. 58])

$$\widehat{\vartheta}_\tau, \vartheta_\tau \to \vartheta \quad \text{strongly in } L^2(0,T;L^2(\Omega)) \equiv L^2(Q). \tag{5.12}$$

From (5.9d) we also have

$$\ln\vartheta_\tau \to \ell \quad \text{weakly in } L^2(Q). \tag{5.13}$$

In consequence, the last two convergences lead to

$$\limsup_{\tau\to 0} \int_Q \vartheta_\tau \ln\vartheta_\tau = \int_Q \vartheta\ell.$$

Then, in view of [4, Exemple 2.3.3, p. 25] all assumptions of [4, Proposition 2.5, p. 27] are fulfiled and we get

$$\ell \in \ln\vartheta \quad \text{in } L^2(Q), \ \text{and therefore} \quad \vartheta > 0 \ \text{ and } \ \ell = \ln\vartheta \quad \text{a.e. in } Q. \tag{5.14}$$

In addition, we can conclude $\xi = \beta(\vartheta)$ by Lemma 3.9.

By recalling Definition 3.5 the difference between the translated and non-translated step approximation can be estimated in the following way

$$\|\vartheta_\tau - \mathcal{T}_{-1}(\vartheta_\tau)\|^2_{L^2(Q)} \leq \tau^2 \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \left\|\frac{\vartheta^i - \vartheta^{i-1}}{\tau^i}\right\|^2_{L^2(\Omega)} \leq \tau^2 \left\|\partial_t\widehat{\vartheta}_\tau\right\|^2_{L^2(Q)} \leq C\tau,$$

Next, using the norm estimate (5.3) and (5.12) we can conclude

$$\mathcal{T}_{-1}(\vartheta_\tau) \to \vartheta \quad \text{strongly in } L^2(0,T;L^2(\Omega)).$$

Now, the assumptions of Lemma 3.8 are fulfilled by $\overline{\pi}_\tau(\mathcal{T}_{-1}(\vartheta_\tau))$ and we obtain

$$\overline{\pi}_\tau(\mathcal{T}_{-1}(\vartheta_\tau)) \to \pi(\vartheta) \quad \text{strongly in } L^2(0,T;L^2(\Omega)). \tag{5.15}$$

Finally, the remaining nonlinear terms (i.e. those related to G, F' and G') can be identified by using the convergences given in (5.11) accounting for our assumptions (2.2b).

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