Master Seminar
Trend to equilibrium in dissipative PDEs

WS 2016/2017

Dr. A. Schlichting (schlichting@iam.uni-bonn.de)

The seminar concerns the long time behavior of *dissipative* PDEs. In physics, one generally thinks of dissipativity as a dissipation of some "energy" associated with the system. Mathematically speaking this energy E acts as a Lyapunov function of the dynamic and it is often possible to derive a Gronwall type inequality of the form $dE/dt \leq -\Phi(E)$ from which one can deduce the long time asymptotic.

1) Energy–energy-dissipation (EED) principle for the *Fokker-Planck* equation of a particle density $f(t,v)$ with $t \in \mathbb{R}_+$ and $v \in \mathbb{R}^d$

$$\partial_t f = \Delta_v f + \nabla_v \cdot (fv).$$

Trend to equilibrium via entropy dissipation: Survey and overview [MV99]

   a) Decay to the thermal equilibrium state [AMTU02, Section 2]

   b) Sobolev inequalities and its consequences [AMTU02, Section 3]

   c) Logarithmic-Sobolev inequalities and hypercontractivity [G75, G93]

2) In the above analysis, a crucial observation consists in rewriting the equation as $\partial_t f = A^* A f$ with $A = \nabla_v$ and the adjoint wrt. to a weighted L^2 inner product. In the next part of the seminar, we want to extend this to operators of *Hörmander* type and f now solves an equation of the form $\partial_t f = A^* A + B$, where B is antisymmetric $B^* = -B$. A first example is the *kinetic Fokker-Planck* equation for a particle density $f(t,x,v)$

$$\partial_t f + v \cdot \nabla_x f - \nabla V(x) \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (fv).$$

Hereby, the function $V : \mathbb{R}^d \to \mathbb{R}$ is a potential ensuring the existence of a equilibrium of finite total density. The equation is kinetic in the sense, that it involves not only position $x \in \Omega \subseteq \mathbb{R}^d$, but also velocity variables $v \in \mathbb{R}^d$. The equation is the prototype of the so called *hypocoercive* phenomenon: The differential operator of the rhs. is degenerate diffusive (only in v) of the form $A^* A$ and the left hand side is conservative satisfying $B^* = -B$ at it describes the trajectories of a classical Hamiltonian dynamical system for the Hamiltonian $H(x,v) = V(x) + |v|^2/2$. The Hamiltonian also defines the stationary state $f_\infty(x,v) = \exp(-H(x,v))$.

a) A refined energy–energy-dissipation principle is still possible if $V \equiv 0$ on a bounded domain Ω to show exponential convergence to equilibrium [CT98]

b) In general by using a local equilibrium it is possible to establish a system of differential inequalities providing the convergence to equilibrium. A crucial ingredient is the use of local equilibrium defined with the help of the Maxwellian $M(v) = \exp(-|v|^2/2)$ and density $\varrho(x) = \int f(t,x,v)\,\mathrm{d}v$ by ρM. The crucial functional is the relative entropy, defined for two densities f, g by $H(f|g) = \iint f \log \frac{f}{g}\,\mathrm{d}x\,\mathrm{d}v$, which allows the additive splitting $H(f|f_\infty) = H(f|\varrho M) + H_x(\varrho|e^{-V})$.

   Heuristics, second derivative of the entropy and system of differential equations [DV01, Section 1-3,6]

c) Nonlinear interpolations and uniform in time hypoelliptic estimates [DV01, Section 4,5]

d) A systematic abstract study exploiting the necessary conditions on the operators A and B from the Hörmander form $L = A^*A + B$ is then investigated [V07, Review] and [V09, Section I.1-4].

e) and can be applied to the Fokker-Planck equation [V09, Section I.6-7]

f) A concise slightly different approach is discussed in [AS16, DMS15].

3) The main motivation of the above analysis is the proof of the trend to equilibrium of the inhomogeneous Boltzmann equation for a particle density $f(t,x,v)$

$$\partial_t f + v \cdot \nabla_x f = Q(f,f).$$

Here $Q(f,f)$ is the Boltzmann collision operator modeling elastic collision of small particles in a gas. The above program can be implemented as follows

a) First, we will investigate the dissipative structure of the homogeneous equation $\partial_t f = Q(f,f)$ [CC92, CC93]

b) and prove therewith the convergence to equilibrium [TV99]

c)–... Trend to equilibrium of the inhomogeneous equation [DV05, V08].

References

[AMTU02] A. Arnold, P. Markowich, G. Toscani, A. Unterreiter, On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations, Commun. Partial Differ. Equations. 26 (2001).

[AS16] A. Arnold, C. Schmeiser, Relative entropies and hypocoercivity revisited, Preprint. (2016) 1–16.

[CC92] E.A. Carlen, M.C. Carvalho, Strict Entropy Production Bounds and Stability of the Rate of Convergence to Equilibrium for the Boltzmann Equation, J. Stat. Phys. (1992).

[CC93] E.A. Carlen, M.C. Carvalho, Entropy production estimates for Boltzmann equations with physically realistic collision kernels, 1994.

[CT98] J.A. Carrillo, G. Toscani, Exponential convergence toward equilibrium for homogeneous Fokker–Planck-type equations, Math. Methods Appl. Sci. 1286 (1998).

[DMS15] J. Dolbeault, C. Mouhot, C. Schmeiser, Hypocoercivity for linear kinetic equations conserving mass, Trans. Am. Math. Soc. 367 (2015) 3807–3828.

[DV01] L. Desvillettes, C. Villani, On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: The linear Fokker-Planck equation, Commun. Pure Appl. Math. 54 (2001) 1–42.

[DV05] L. Desvillettes, C. Villani, On the trend to global equilibrium for spatially inhomogeneous kinetic systems: The Boltzmann equation, Invent. Math. 159 (2005).

[G75] L. Gross, Logarithmic Sobolev Inequalities, Am. J. Math. 97 (1975) 1061.

[G93] L. Gross, Logarithmic Sobolev inequalities and contractivity properties of semigroups, in: Dirichlet Forms, Springer Berlin Heidelberg, 1993: pp. 54–88.

[MV99] P.A. Markowich, C. Villani, On the trend to equilibrium for the Fokker-Planck equation: an interplay between physics and functional analysis, (1999) 1–24.

[N73] E. Nelson, The free Markoff field, J. Funct. Anal. 12 (1973) 211–227.

[TV99] G. Toscani, C. Villani, Sharp Entropy Dissipation Bounds and Explicit Rate of Trend to Equilibrium for the Spatially Homogeneous Boltzmann Equation, Commun. Math. Phys. 203 (1999) 667–706.

[V07] C. Villani, Hypocoercive diffusion operators. Proc. Int. Congr. Math. Madrid, August 22–30, 2006, European Mathematical Society Publishing House, 2007.

[V08] C. Villani, Entropy production and convergence to equilibrium, Lect. Notes Math. 1916 (2008) 1–70.

[V09] C. Villani, Hypocoercivity, Mem. Am. Math. Soc. 202 (2009).