Master Seminar: Phase separation and interface evolution

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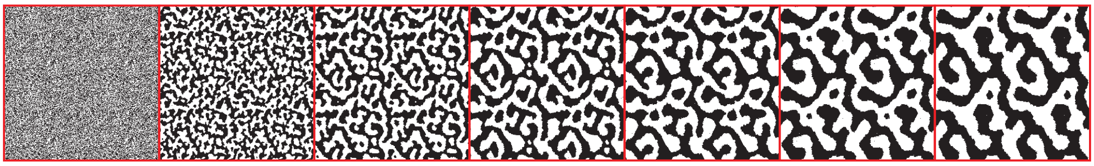

The objective of this seminar is to describe the process in which two components of a binary fluid in a domain Ω separate and form pure phases. If $c : \Omega \to [-1, 1]$ indicates the concentration of the phases ± 1, then the energy of c is given by

$$E_\varepsilon(c) = \int_\Omega \underbrace{\frac{\varepsilon}{2} |\nabla c|^2}_{\text{interface}} + \underbrace{\frac{1}{\varepsilon} W(c)}_{\text{bulk}} \, \mathrm{d}x, \qquad \text{with } W : \mathbb{R} \to [0, \infty) \text{ smooth and } W(c) = 0 \text{ iff } c = \pm 1.$$

The interface part of the energy penalizes transitions between the phases, whereas the bulk part prefers the pure phases ± 1. The parameter $\varepsilon > 0$ determines the width of the transitions. The energy E_ε shows non-trivial minimizers if one fixes the phase fraction, i.e. considers the constraint variational problem

$$\min \left\{ E_\varepsilon(c) \;\middle|\; c : \Omega \to \mathbb{R} \text{ s.t. } \frac{1}{|\Omega|} \int_\Omega c \, \mathrm{d}x = m \in (-1, 1) \right\}.$$

The first two seminar topics consider this minimization problem [M87, LM89].

The Cahn-Hilliard equation is a dynamics reducing the energy during time and keeping the phase fraction constantly:

$$\partial_t c = -\Delta \left(\varepsilon \Delta c - \tfrac{1}{\varepsilon} W'(c) \right).$$

The next step is to investigate the so called sharp interface limit for $\varepsilon \to 0$ of the fourth-order PDE, which will be first established by formal asymptotics [P89]. A rigorous derivation uses linearization, spectral analysis and approximation [ABC94, C94].

Afterwards, the limit model, which is the Mullins-Sekerka dynamics, will be studied [N04].

One can physical motivate the appearance of an additional term in the Cahn-Hilliard equation, which is of the form

$$\partial_t c = -\nabla \cdot M(c) \nabla \left(\varepsilon \Delta c - \tfrac{1}{\varepsilon} W'(c) \right),$$

where $M(c) = 1 - c^2$ is called mobility and degenerates as $c \to \pm 1$. Again, by formal asymptotics as $\varepsilon \to 0$, it is possible to find a limiting equation [CENC96].

The Cahn-Hilliard models considered so far share the property that in the long run, the dynamic converges to the minimizer of the Ginzburg-Landau energy for a given phase fraction. The question, whether this is true for any dynamic reducing the Ginzburg-Landau energy during its evolution, is investigated in the last talk [NCP91].

First Seminar Talk: Monday, 13th October, 16:15 in room 2.040

[ABC94] N.D. Alikakos, P.W. Bates, X. Chen, Convergence of the Cahn-Hilliard equation to the Hele-Shaw model, Arch. Ration. Mech. Anal. 128 (1994) 165-205.

[CENC96] J.W. Cahn, C.M. Elliott, A. Novick-Cohen, *The Cahn–Hilliard equation with a concentration dependent mobility: motion by minus the Laplacian of the mean curvature*, Eur. J. Appl. Math. 7 (1996).

[C94] X. Chen, *Spectrum for the Allen-Cahn, Cahn-Hillard, and phase-field equations for generic interfaces*, Commun. Partial Differ. Equations. 19 (1994) 1371–1395.

[G13] H. Garcke, *Curvature Driven Interface Evolution*, Jahresbericht der DMV. 115 (2013) 63–100.

[LM89] S. Luckhaus, L. Modica, *The Gibbs-Thompson relation within the gradient theory of phase transitions*, Arch. Ration. Mech. Anal. 107 (1989).

[M87] L. Modica, *The gradient theory of phase transitions and the minimal interface criterion*, Arch. Ration. Mech. Anal. 98 (1987).

[N04] B. Niethammer, *Averaging methods for phase transition problems*, Lecture Notes, Rome, 2004.

[NCP91] A. Novick-Cohen, R.L. Pego, *Stable Patterns in a Viscous Diffusion Equation*, Trans. Am. Math. Soc. 324 (1991).

[P89] R.L. Pego, Front Migration in the Nonlinear Cahn-Hilliard Equation, Proc. R. Soc. A: 422 (1989) 261–278.